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**Is the graphic calculator a useful mediating
tool for students in the early stages of
forming a concept of a variable?**

A Thesis for the
Degree of Doctor of Philosophy (PhD)

by Jennifer Anne Gage MA (Cantab), BA (Open)

The Open University
Centre for Mathematics Education

23 July 2004

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afloat.

This thesis is dedicated to the memory of my mother
Beryl Gage (1927-1997).

ABSTRACT

The graphic calculator offers an environment in which children can start to understand basic algebra. It acts as both a mediating physical tool and psychological sign in the sense Vygotsky described, shaping the higher mental processes of the students. The combination of a graphic calculator and two students is theorised as a zone of proximal development, a term used by Vygotsky to indicate the potential students have to achieve more when supported than they could do alone. The graphic calculator also acts as a focus for reflective discussion, providing students with language to enable them to articulate their ideas, and a locus for trying out those ideas. The immediate feedback provided enables students to challenge misconceptions they already hold, so enabling them to develop conceptions that are more appropriate.

The graphic calculator forms a learning environment by providing a model for a variable that is concrete and easily understood by even quite young children. The stores of the calculator are labelled with alphabetic letters, and so can be thought of as boxes into which numbers can be put. These stores can then be operated on in the same way as an algebraic variable. Although this model is not sufficient to explain a variable as a number that can change continuously, it is quite adequate to help children understand the concept of a variable up to the stage of a generalised number.

Three case studies and a survey, using the graphic calculator model of a variable and teaching materials designed to exploit its affordances, are discussed in this thesis.

Instances were found of students making cognitive gains as a result. Statistical evidence indicating that the students improved both their understanding of the nature of variables, and their skills in working with simple algebraic expressions is also given.

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CHAPTER 1 INTRODUCTION

1.1 STATEMENT OF THE PROBLEM

In answer to a question asking what the a and b in an expression like $4a + 3b + 2a$ might mean, one student (aged 12 or 13 years, and in Year 8 in a UK secondary school) wrote: “I think that a and b are only letters that don’t mean anything.” A second student of the same age wrote: “ a and b are just fancy things at the end of a sum.” A third just put: “?” These students took part in a survey carried out during the classroom research reported in this thesis. They had studied some algebra previously, but had yet to see any meaning in the letters used. The findings of this research would suggest that this is not unusual in students of this age, despite the fact that they may well have studied algebraic topics several times during the previous year or two in their mathematics lessons.

The research described in this thesis originated in my desire to help students like these to secure some understanding of what the mysterious letters used in algebra mean. In this thesis, I argue that use of the graphic calculator can enable students to improve their understanding of how letters are used in algebra, and their skill in working with algebraic expressions. I argue further that the calculator can facilitate the remediation of some common misconceptions. In support of these arguments, an analysis of data collected during classroom studies is described. These studies were carried out in five different schools, with students from four different year groups.

In this thesis, the role of the graphic calculator as a mediating tool for students in the early stages of working with algebraic variables is considered. The concept of the graphic calculator as a mediating tool for learning is based on Vygotsky's concept of mediation: that the tools used to assist learning will structure that learning. The main research question I wished to explore was:

- Is the graphic calculator a useful mediating tool for students in the early stages of forming a concept of a variable?

Sub-questions I also wished to investigate were:

- Does the model of a variable provided by the graphic calculator mediate successfully between students' initial interpretations of letters and an interpretation which will help their progress in algebra?
- If graphic calculator use proves helpful, what are the attributes of the graphic calculator which make it a suitable tool for students learning algebra?

In my research, I carried out three case studies and a survey to investigate these questions. I looked at how students worked with algebraic expressions, to see if and how the graphic calculator might facilitate their understanding of what letters mean and how they are used. From analyses of the data collected in these studies, I concluded that using the graphic calculator did indeed promote understanding. I also felt that it provided a means of bridging the gap between the ideas students brought with them from everyday life and earlier school experiences and the knowledge they needed if they were to make significant progress in algebra.

Variables can be modelled with a graphic calculator using its 26 stores which are labelled with the letters of the alphabet. The students in my classroom studies put numbers into the

stores, and then operated on these stores in the same way that operations on a variable are carried out. This provided the letters used in algebra with meaning for the students, encouraging them to think of the letters as stores for numbers. It also helped them to realise that the letter used is arbitrary: that a given letter can represent any number, and a given number can be represented by any letter. Replicating 'screensnaps'¹ (graphic calculator screens) helped the students to begin to internalise common algebraic conventions, and to start learning the syntax of algebra.

Data from the classroom studies were examined to see if there was any evidence of cognitive change as a result of the teaching method. Examples of cognitive change in students were found, and these are discussed in detail, as are examples where cognitive change failed to occur. The data were also analysed to see if they supported the proposition that the students' understanding of how to use letters improved, and whether their skill in answering algebraic questions increased during that process. It was found that students made progress in both of these, particularly the younger students and those whose previous understanding of algebraic expressions was least secure. Using the data collected in the case studies and survey, I concluded that the graphic calculator could be used effectively to promote learning.

This research is grounded in Vygotsky's theories, particularly that of the mediation of tools and signs in the development of children's higher mental functions. Vygotsky's theories were conceived against the background of the tumultuous changes the USSR experienced

¹ The word 'screensnap' to denote graphic calculator screens used in this type of work was coined by Alan Graham (Graham, 1998: 22).

in the 1920s. He, like many others, wanted to improve education in the USSR very rapidly for as many children as possible. This led him to suggest many different theorisations of the learning process, which have attracted considerable attention in the last two or three decades since his work became more generally known in the West. His theorisations have provided a foundation for my own theoretical position. In particular, I have used his theory of the mediation of tools and signs in the formation of concepts to theorise how the graphic calculator might mitigate some of the difficulties students often experience in their early encounters with algebra.

I have further formulated a new theoretical model to explain how this might occur.

Vygotsky introduced the concept of the zone of proximal development (ZPD). He defined this as what a student can accomplish when assisted by a teacher or more able peer, that is over and above what s/he might be able to accomplish unaided. In my theoretical model, I suggest that the graphic calculator, when used by a pair of students, is part of a ZPD in which both students are enabled to reach a higher level of understanding than would otherwise be the case.

The misconceptions students brought with them before doing the classroom studies, and those they still held afterwards, are also examined. It is a fallacy to think that students come to a new area of learning like 'blank slates': certainly it was apparent from this research that the students who participated in the case studies and survey made many conjectures about the nature of algebraic variables on the basis of previous learning both in mathematics and elsewhere. Some of their misconceptions were susceptible to the graphic calculator teaching modules, others less so. In addition to ideas which the students brought

with them, some students were also found to develop new misconceptions which appeared to result from their misunderstanding things said to them in class. This is also investigated.

1.2 ORIGINS OF THIS RESEARCH

This research has developed through the interaction of three different strands. The first strand originated in my work as a teacher of mathematics in three English secondary schools over a period of 15 years. At the time I started this research at the beginning of 1998, I had been teaching for some 12 years. I felt that I had become a relatively successful classroom teacher, and yet I was concerned that throughout that time I had failed some of my students. These students were bright, articulate and keen to learn, yet found it difficult to make any sense of algebra despite coping well enough with other mathematical topics. They could see a purpose for arithmetic and statistical work; diagrams and mathematical software helped them to see what was happening in geometrical problems. In algebra, however, they had no idea what the equations and expressions meant, and were only able to answer questions by following rules learnt by rote. Such rote learning meant that they did not recognise when they were using these rules inappropriately, and this often led to errors.

A turning-point came during the spring term of 1998 with a Year 10 Higher Level GCSE group², who were very high achievers, both mathematically and otherwise. I was looking for ways to challenge them so that they acquired a deeper understanding, rather than simply learning yet more techniques. To this end, I introduced the students to graphic

² Year 10 students are aged 14-15 years in English schools. The General Certificate of Secondary Education is an examination sat by 16 year-olds in England and Wales, at the end of Year 11.

calculators as tools to facilitate an exploration of functions represented as formulae, graphs and tables. Some very exciting work resulted from this, with many of the students showing remarkable insights. At this point, I realised two things: that the lack of understanding of algebraic expressions so common even among bright students was not inevitable, but that using unfamiliar approaches like this involved a degree of risk.

We were all affected by the changing emotional temperature which accompanied this project. The students lurched from acute anxiety to the enjoyment of new discovery. “[It was] really scary initially” one said, while another told me “[I] didn’t know what to do – panic!” It was high risk work for all of us. They were new to the instruments, new to the mathematical ideas, with a new teacher with high expectations. I was afraid my expectations of them, the graphic calculators, and, indeed, myself, were too great (Gage, 1999a).

I knew the risk had paid off, however, when I heard a voice ringing across the classroom: “Oh! Wow! I never knew...!” This was a key moment, when it became apparent that the students’ anxiety was being transformed by the excitement they felt at creating their own mathematics in an area which was quite new to them. Comments made by students afterwards included: “the research [was] really enjoyable and the trial and error was a strong point”. Another student said that she felt “more self-motivated and so learnt more”, while a third said that: “[it] had its own momentum – it all fell into place”. Yet another told me: “[I] liked owning my own discoveries”. The remark that best exemplified my hopes in doing this project with them was:

My investigation from the start kept leading me onto new ideas... I feel I have learnt an incredible amount about a subject I never knew before.

I could see that algebraic topics did not need to be full of tedious exercises in which students implemented poorly understood rules. One significant result of this project was that I realised how important it was for students to retain ownership of their work, and to make their own discoveries. It was also clear that if I was going to use this kind of teaching method in the future, I needed to reduce the initial level of uncertainty.

This was the point at which the second strand started to interact with this practical classroom experience. By then, I had become aware of the radical constructivist paradigm, particularly as expressed by von Glasersfeld (1991; 1995). Reading his work while I was engaged in the project described above made me think about what sort of teacher I wanted to be: I knew I wanted to enable my students to construct their own learning more, and to lead from the front less. I wanted them to feel excited about learning new things, but I did not want them to feel quite so anxious about the outcomes as the Year 10 students did at the start of their investigations.

I felt I needed to investigate the graphic calculator as a tool to help students bring together graphic and algebraic models of functions, as in the Year 10 project described above, and therefore decided to see if I could design a research study in which students would use the graphic calculator as a microworld in which to build up knowledge. This would use many of the features of the Year 10 study, but this time I wanted to help the students to feel more supported during the early stages, without taking away their excitement at making their own discoveries.

I then encountered a paper by Margot Berger (1998) which stimulated my thinking about graphic calculator research more generally. A particular claim she made, that there was a

scarcity of research about *how* the graphic calculator might function as a tool for learning, encouraged me to move the focus of my thoughts away from the details of the algebraic teaching to the way in which the graphic calculator could mediate between the students and the mathematics. She had based her research on Vygotsky's work on the mediation of tools, a theory which seemed particularly interesting in the light of my work with the Year 10 students. One particular quotation she cited set in motion the research project that followed:

If one changes the tools of thinking available to a child, his mind will have a radically different structure. (Berg, 1970: 164, cited in the *Afterword*, Vygotsky, 1978: 126)

This sentence stayed with me for a long time. For a while, I thought this encapsulated what I wanted to do in my research, until I realised that it would be extremely difficult to demonstrate that a child's mind had a 'radically different structure' as a result of using the graphic calculator.

However, Berger also emphasised the difference between 'amplification' and 'cognitive reorganisation' as metaphors for how the graphic calculator might act as a tool for learning (cf. Pea, 1985, 1987). She carried out a study, looking for cognitive reorganisation, but failed to find much evidence for it. As it seemed to me that this was due more to the limitations of her experimental method than to the fact that such evidence could not be found, I decided to do some studies of my own to see if evidence could be found to support either or both of these as a metaphor to describe how the graphic calculator might function as a tool for learning. As a result of my experience with the Year 10 group, I was sure that it would be possible to find evidence for cognitive change caused by using the graphic calculator.

The third strand which brought my research topic into even closer focus, occurred in the spring term of 1999, when I started working during the lunch hour with a particular student, Sally³, who was preparing for the forthcoming Key Stage 3 SATs⁴ examinations. Sally found all mathematics difficult, and algebra was still virtually a closed book to her. We started by looking at a revision paper her class had done during the previous week, which contained equations of varying degrees of difficulty to solve. Gradually we worked backwards through less and less demanding questions as I tried to find a level at which Sally was comfortable. Eventually, I asked her what $2x - x$ was. “2!” she replied, thankful at last to find a question she knew she could answer (this is discussed in more detail in Gage, 2002a).

At this point, we went right back to basic ideas about what the letters in simple expressions such as $2x$ meant. I used a model for a variable which I had discovered in Graham’s work (Graham, 1998; Graham and Thomas, 1998). He suggested that the stores of the graphic calculator, which are labelled with letters, could be used to provide students with a physical model of a variable, and could also be used to show them how to work with variables. This was ideal for Sally. She could understand the idea of boxes or stores with numbers in them, and that you could add to them, subtract from them, multiply them by other numbers, and so on. We started with pictures of boxes with numbers (both known and unknown) in them and from there graduated to using the graphic calculator. Sally found the calculator method straightforward to understand, and she operated the calculator easily. We started to make progress, with Sally gaining in confidence.

³ All student names used in this thesis are pseudonyms.

⁴ Standard Assessment Tests set at the end of Key Stages in English schools. This particular girl was aged 13 or 14 and was in Year 9, which is the end of Key Stage 3.

The success of this approach led to a pilot study with Sally's class during the summer term of 1999. I was still searching for a specific mathematical focus for my research, although I knew by this stage that I wanted to look at using graphic calculators as mediating tools for students to use in learning algebra. The work with Sally, followed by the pilot study, brought home to me how crucial it is to students' further progress that they gain an early understanding of how letters should be interpreted and what simple expressions mean. I therefore focused my research question on this issue.

The question I decided I wanted to answer was two-fold. Firstly, I wanted to find out if and how Graham's model might mediate between the ideas students brought with them when they began to study algebra and the more formal instruction they encountered in school. Secondly, I wanted to see if and how the use of a physical tool, such as the graphic calculator, would help students in developing understanding of basic algebraic operations.

1.3 OUTLINE OF THIS THESIS

This thesis discusses a theoretical model for how the graphic calculator can act as a mediator between a student and the concept of a variable, and an evaluation of this model based on three case studies and a survey. The graphic calculator model of a variable, devised by Graham, was further developed by Graham and Thomas (Graham, 1998; Graham and Thomas, 1998, 2000a). They used the labelled stores of the graphic calculator as a physical instantiation for variables in work they carried out with 14 to 15 year-old students in the UK and New Zealand. The calculator stores can be operated on in the same way as the letters used in algebraic expressions, giving students practice and a means to understand how letters are used in algebra. Their model is extended in this thesis into a theoretical model grounded in Vygotsky's theories.

The classroom method used in the studies reported in this thesis involved students working together in pairs with a graphic calculator between them to enable them to make and test conjectures about how letters used in algebraic expressions function. Students were asked to copy 'screensnaps' (graphic calculator screens) requiring the evaluation of simple expressions, or checking which of various alternative expressions were equivalent to each other. The exercises the students did introduced them to the main conventions of algebraic expressions, such as omitting the multiplication sign in products, and using the '/' sign for division. Many students were quite surprised to discover that, for instance, $8B$ means 8 multiplied by the number in the B store. Many expected that this would equal 84, if 4 had been put into the B store, or even 82^5 . The screensnap shown in Figure 1 illustrates this example:

Figure 1: Screensnap showing how students can test their ideas about the meaning of expressions like $8B$ using the graphic calculator

4→B	
8B	4
	32

Even a simple example like this shows how the graphic calculator can help students.

Firstly, the calculator provides a physical model of a variable through the action of putting a number into a store. Every time the students put a number into a store, the screen display reinforces the idea that the letter represents a number. Secondly, the graphic calculator allows students to test out their ideas. Finding that $8B$ is neither 84 nor 82 challenges the students to rethink their understanding of the expression $8B$: it soon becomes obvious that

⁵ Equating a with 1, b with 2, c with 3, ..., was a common initial misconception among the students.

the value in the B store is to be multiplied by the 8. As the number of examples like this grows, the student gradually internalises the correct interpretation of such expressions.

Rather than trying to use rules which they do not understand, students can verify their conjectures using the graphic calculator to tell them if they are correct. For example, consider the question cited in the opening sentences of this thesis: $4a + 3b + 2a =$. Confronted with this question, a student might be very unsure what the answer should be. The data showed that many of them tried to add up the coefficients first, a familiar operation, giving a value of 9. The question then was what to do about the letters. Some students just ignored them, putting down '9' as their answer. Some decided that an 'a' and a 'b' and another 'a' gave 'd'; others that 'a' and 'b' should be put together to make 'c' or 'ab'. Experimenting with the graphic calculator meant that students could try all these (plus anything else they thought of) to find out which, if any, were correct. The screensnap below shows how they might proceed:

Figure 2: Screensnap illustrating how students might go about testing equivalent expressions for $4a + 3b + 2a$

4A+3B+2A	
9D	39
9C	27
9AB	63

In the teaching method used in this research, learning took place within pairs and groups of students as much as on an individual basis. Discussions between students gave them opportunities to elaborate their thoughts and to clarify new insights. The graphic calculator also provided language with which students could discuss their ideas. The role of the

teacher is then to challenge the students with questions and examples which will open up their thinking, and to involve the whole class in discussion when an important point occurs.

1.3.1 First part of the thesis

This thesis can be divided into two major parts. The first part establishes the ground on which the rest depends, and is described in this section. It starts with a description of the relevant literature, Chapter 2: *Review of the literature*, which provides a basis for the theoretical model developed in the second part of the thesis, and for the varieties of analysis reported there. This is followed by Chapter 3: *Research methodology and methods* in which the mixed methodology used in the classroom research is justified, and the methods of data collection and analysis used in the case studies and survey described.

The *Review of the literature* commences with a discussion of theories of learning in mathematics. The position taken in this thesis is that mathematics is constructed by human endeavour, rather than already in existence and ‘out there’ waiting to be discovered. Constructivist learning theories are considered, including those of Vygotsky. The theoretical model developed later in the thesis depends on certain of Vygotsky’s theoretical foci which are reviewed in this chapter. These include:

- Mediation of tools – the use of appropriate tools to carry out practical activities.
- Mediation of signs – the role of psychological ‘signs’ in enabling humans to develop higher mental functions, for instance, putting a knot in a handkerchief to aid the memory.
- Concept formation – occurring through the interaction of everyday ideas (such as putting numbers into stores) and taught ‘scientific’ ideas (such as an algebraic variable).

- Key role of speech – the facilitation of such interaction by language and discussion.
- Zone of proximal development – the gap between what a student can do unaided and what s/he can do when supported.

Vygotsky's model of concept formation is referred to frequently in this thesis. He considered that stable concepts are formed when everyday and scientific (or school) knowledge interact. In my theorisation, the graphic calculator acts as a mediating tool by providing a locus in which the everyday and scientific forms of knowledge that students have about variables can combine. More recent ideas on instrumentation, the process by which an artefact such as a calculator becomes a useful tool, are also considered. These processes are all facilitated by discussion between pairs of students, and larger groups, supported as necessary by the teacher.

As well as looking at the mediating action of practical tools, Vygotsky considered the mediating action of psychological signs⁶. Examples of such signs are language, both written and verbal, arithmetic, and algebra. Signs are equivalent to tools, but enable us to amplify our mental capabilities as opposed to our physical capabilities. They are invented by us, to enable us to extend our higher mental functions. The graphic calculator also acts as such a sign by changing the way the students think about algebraic variables: it increases their capacity to work with them in the same way that words increase our capacity to remember objects.

Such changes are not one-way, however. Vygotsky saw both mediating tools and mediating signs as causes of dialectical change: we modify our environment, using physical tools and psychological signs, and such modification changes us. Changing the learning environment can lead to cognitive development in students, the exact nature of which is at least in part a consequence of the tools and signs used. Evidence is presented in this thesis to show that the graphic calculator does indeed cause cognitive development in students, and thus acts as a tool and a sign in this way.

In *Research methodology and methods*, there is a detailed description of the classroom studies described in this thesis, which were carried out between the summer (April-July) terms of 1999 and 2002. These consisted of three case studies and a survey, designed to test the proposition that using the graphic calculator would help students in the 10 to 14 year age range in developing an understanding of how letters are used in algebra. In all, some 400 students were involved from five different schools, across four different year groups. A range of different data forms were collected, including audio and videotapes of classroom discussions, interview data and questionnaires. These data were used with the literature survey to develop the theoretical model of how the graphic calculator acts in the learning environment. The data were also used to judge the extent to which cognitive change occurred, and students made progress in their understanding and skills.

⁶ According to Vygotsky, a sign system, such as language, writing, or a number system, is created by a society over the course of human history and changes with the nature of the society and the level of its cultural development. "The use of signs leads humans to a specific structure of behavior that breaks away from biological development and creates new forms of a culturally-based psychological process." (Vygotsky, 1978: 40).

1.3.2 Second part of the thesis

In the second major part of this thesis, the findings of the classroom studies are discussed. The findings are presented thematically, with four major themes explored: the graphic calculator as a cognitive technology (Chapter 4); cognitive change in the students mediated by the graphic calculator (Chapter 5); development facilitated by the graphic calculator in the students' understanding and skills (Chapter 6); and misconceptions shown by the students, and the effect of the graphic calculator on these (Chapter 7). The first of these themes is a theoretical analysis of the way in which the graphic calculator mediated in the learning process as both tool and sign. The other three discuss the effects of this mediation.

Before presenting these themes in a little more detail, it is worth reviewing the features of the graphic calculator that make it a suitable instrument for use in the classroom with students in the 10-14 year age group. First, the small screen of the graphic calculator allows privacy to the student(s) working with a particular instrument: no one else needs to see what is on a student's screen if they would rather not. This means they can try out any conjectures they like without having to explain them, or risk embarrassment, if others disagree. Secondly, the graphic calculator gives immediate feedback. If students try to replicate a screen, they know straight away if they have done it correctly or not. The graphic calculator provides a private learning environment in which students can discuss their ideas, verbalising their insights, and negotiating a consensus where there is disagreement. Thirdly, a class set of graphic calculators can be available to a teacher during every lesson, so that the calculators are always there to support the students. Using computers instead may be more difficult, since gaining access to them on a regular basis with a whole class is often problematic.

All of the above features of the graphic calculator are reasons why its use might be advantageous in the classroom. However, there is more to it than that. The graphic calculator is a technological tool, and these form a significant part of students' cultural backgrounds today. Consider, for instance, students' use of mobile phones for sending text messages, and of games machines for recreation. Wertsch's (1988) analysis of Vygotsky's work identified higher mental functioning mediated by socioculturally evolved tools and signs as a major strand in Vygotsky's theory. In this thesis, the graphic calculator is identified as a sociocultural tool. The graphic calculator acts as a mediating tool: students carry out practical activities using the calculator. It also acts as a mediating sign: students' understanding is changed by the results of their actions with the calculator. This theme is then extended to a consideration of metaphors for how such tools might mediate learning.

Chapter 4: *The graphic calculator: Mediating in a learning environment* presents a detailed discussion of this theme.

According to Vygotsky, the whole point of such sociocultural tools is that they enable higher mental functioning, and this is discussed further in Chapter 5: *Evidence of cognitive change*. The interpretation of cognitive change used in this chapter is defined, and then examples are given. These examples show that the graphic calculator was indeed instrumental in causing such change on a number of occasions. These are compared with other examples where cognitive change failed to occur, and reasons for this failure are discussed. Transcripts from the classroom case studies carried out are used to support these examples. Chapter 6: *Developments in students' understanding and skills* presents an analysis of data collected to demonstrate students' progress both in understanding and in skill. It was found that students with little previous experience and/or understanding of algebra made excellent progress both in understanding and in skill. The final theme is

presented in Chapter 7: *Misconceptions*, which is an investigation into the misconceptions students brought with them prior to doing this work, and the effect of the graphic calculator on these misconceptions. This chapter also includes an examination of misconceptions not initially obvious but which became apparent during the teaching process.

Following these two major sections of the thesis, Chapter 8: *Conclusions and recommendations* summarises the thesis, presents its major conclusions and gives recommendations for further research. This is then followed by the *Annexes* and the *References*.

CHAPTER 2 REVIEW OF THE LITERATURE

2.1 INTRODUCTION

In this thesis, I wish to address the following research question:

- Is the graphic calculator a useful mediating tool for students in the early stages of learning algebra?

I argue that it is such a tool for the following reasons:

- The graphic calculator is a suitable cultural tool and cognitive technology for 10-14 year-old students.
- The graphic calculator provides an interface between the student and the algebra by providing both a model of a variable and a cognitive tool.
- The graphic calculator used by a pair of students can contribute to a zone of proximal development (ZPD).

The graphic calculator has the potential to enable learning to occur in students working with it on algebraic tasks which exploit both the model of a variable provided by the calculator, and its attributes as a cognitive tool. The model of how the calculator participates in forming a ZPD is heavily dependent on Vygotsky's theory of the mediating action of tools and his social theory of learning.

This chapter reviews the literature in relevant areas. It starts by briefly considering the nature of mathematics and mathematics education (section 2.2). In section 2.3, the

constructivist theories of learning of Vygotsky and Piaget are discussed. Although the major theoretical positions of this thesis are dependent on the work of Vygotsky, it is impossible to ignore Piaget's contribution to theories of learning, particularly to the development of the various forms of constructivism. Vygotsky's work is then examined in more detail in section 2.4, together with developments of his work by the socioculturalists who followed him. In section 2.5, the mediating role of tools is considered. Section 2.6 then moves on to consider issues in the teaching and learning of algebra, looking at children's difficulties in learning algebra, and some responses to these. In section 2.7, the contribution to the teaching and learning of algebra that the graphic calculator can make is considered. The final section, 2.8, summarises this chapter.

2.2 EPISTEMOLOGY OF MATHEMATICS UNDERLYING THIS THESIS

Vergnaud (1990) argued that all mathematics education research and teaching has an implicit underlying epistemology, even if this is not acknowledged. Epistemology is concerned with questions such as "What is knowledge?" and "How is knowledge to be acquired?" A focus on such questions is fundamental to an understanding of how new technologies enable knowledge to be constructed:

... new technologies – *all* technologies – inevitably alter how knowledge is constructed and what it means to any individual. This is as true for the computer as it is for the pencil, but the newness of the computer forces our recognition of the fact. There is no such thing as unmediated description: knowledge acquired through new tools is new knowledge. (Noss and Hoyles, 1996: 106)

The most common epistemology of mathematics education has been Platonic:

"mathematical activity consists in the discovery of timeless truths ... independent of culture, and ... is mainly a matter of logical reasoning" (Vergnaud, 1990: 28). In this

epistemology, mathematical objects exist independently of us: they are 'given', and are unchanging in their nature.

However, there is a long tradition of viewing knowledge as a human construct, dating back to Giambattista Vico (1668-1744) in the eighteenth century, and earlier. Contrary to the dominant Platonic and Cartesian view of mathematics, that it exists *a priori* outside the individual, Vico believed that:

... arithmetic, geometry, and their offspring, mechanics, are human faculties, since in them we demonstrate a truth because we make it. (Vico, 1710/1988: 94).

We can see the origins of the constructivist view of education here (cf. von Glasersfeld, 1995). Vico sought for an alternative to Cartesianism, which claimed that knowledge is formed and exists outside of the human mind, instead arguing that the construction of all rational knowledge is human:

... we do not just discover the truth, but make it. ... the physical things will be true only for whoever has made them, just as geometrical [proofs] are true for men just because men make them. (Vico, 1710/1988: 104)

During the latter half of the twentieth century, this alternative epistemology has gained ground: mathematics is viewed as a human construct, which is therefore inherently fallible (e.g. Ernest, 1994; Hersh, 1994; Triadafillidis, 1998). Proponents argue that it can be changed at any time according to the circumstances of the human minds constructing it; others maintain that it is socially negotiable (Lerman, 1996a). Mathematics is what mathematicians do; it is the tool of scientific enquiry (Taylor, 1996). This is the view of mathematics and mathematics education which underpins the research described in this thesis. An active, constructivist view of how children learn is assumed throughout.

2.3 CONSTRUCTIVIST THEORIES OF LEARNING

2.3.1 Vygotsky and Piaget

Both Vygotsky and Piaget created theories of learning which have been of considerable relevance to the theoretical underpinning of mathematics education over the last few decades. They are among the most significant names of the twentieth century in the field of the psychology of educational development, and their work has been immensely important for the construction of current learning theories. Both had what would now be termed a constructivist epistemology, believing that human knowledge is constructed, not discovered. Whereas Piaget can be considered the forerunner of the constructivism of the late twentieth century, however, those working in a Vygotskian tradition tend to emphasise other aspects of his thought.

Piaget was active from the 1920s throughout the twentieth century, whereas Vygotsky's contributions in the areas of psychology and educational theory were mainly written between 1924 and his death in 1934. In 1936, Stalin suppressed Vygotsky's work, and it only became accessible again in 1956 in the USSR, and in 1962 in the West when the first translations into English were made. Much of his published work has only become available since the 1980s, so he has been far more influential in the development of late twentieth century thought than might otherwise have been the case. Neither Piaget nor Vygotsky produced a unified body of work, continuing to formulate ideas as they worked on them, but major themes which recurred or were developed throughout their work can be identified.

Vygotsky and Piaget both knew each other's work, and valued each other's contribution, having more in common than is often supposed (Brown, Metz and Campione, 1996a; Smith, 1996). Both had a view of the nature of knowledge and learning which is constructivist: they saw learning as actively constructed by the child, purposeful, adaptive, and relevant to the child's situation (Gruber and Voneche, 1977: xxiif; Vygotsky, 1987: 170). Vygotsky has not been as strongly identified with the recent constructivist movement as Piaget, who has been identified as a direct precursor of radical constructivism (von Glasersfeld, 1995), but an examination of Vygotsky's writings shows that he also believed that children construct their knowledge, rather than receiving it passively. In his discussion of the formation of concepts, he emphasised repeatedly that:

... scientific concepts are not simply acquired or memorized by the child and assimilated by his memory but arise and are formed through an extraordinary effort of his own thought (Vygotsky, 1987: 176, original italics)

Although Piaget and Vygotsky both believed that knowledge is actively constructed, their explanations for this were different. Piaget claimed that development is the result of disequilibrium between the child and her/his environment. Resolution of this disequilibrium (adaptation) leads to the child acquiring new knowledge. Vygotsky, on the other hand, argued that development is essentially a social phenomenon, asserting that "higher psychological processes" have their origin in social or cultural processes (van der Veer and Valsiner, 1994: 138).

An interpersonal process is transformed into an intrapersonal one. Every function in the child's cultural development appears twice: first, on the social level, and after, on the individual level; first, between people (interpsychological), and then inside the child (intrapsychological)¹. ... All the higher functions originate as actual relations between human individuals. (Vygotsky, 1978: 57, original italics)

¹ The words translated here as "interpsychological" and "intrapsychological" are translated by some writers as "intermental" and "intramental", e.g. Minick (1996).

Vygotsky further argued that socially evolved and socially organised cultural tools, particularly language, mediate all learning (Wertsch and Tulviste, 1996). It is a common misconception that Piaget ignored such social processes, but this is too facile:

... the human being is immersed right from birth in a social environment which affects him just as much as his physical environment ... (Piaget, 1947/1950: 156).

Indeed, Piaget claimed that “conceptual thought is collective thought obeying common laws” (*The Construction of Reality in the Child*, 1937/54, cited in Gruber and Voneche, 1977: 279). However, where Vygotsky believed the social environment of the child was all-important in her/his development, Piaget saw it as one influence amongst many, as often arguing against the significance of social factors as for them.

2.3.2 Later constructivist paradigms

Several epistemological positions given the label ‘constructivist’ were advanced during the 1980s and 1990s. Significant among these were radical constructivism (e.g. von Glasersfeld, 1990; 1991; 1995) and, in reaction to it, socioconstructivism (e.g. Bauersfeld, 1992; Cobb, 1991; Ernest, 1991). Such epistemologies have replaced behaviourist models of rote learning, and have had considerable impact on teaching and learning during the last two decades (Zevenbergen, 1996).

Radical constructivists based their position on that of Piaget, that knowledge acquisition has an adaptive function for the individual:

... knowledge is not passively received but built up by the cognising subject; the function of cognition is adaptive and serves the organisation of the experiential world, not the discovery of ontological reality. (von Glasersfeld, 1995: 18)

The individual learns something, not because it is there to be learnt, but because it helps them to cope with the world they experience. In the mathematics classroom, students need to find their own way to solve problems, hence building up knowledge which helps them to organise what they experience. ‘Reality’ is thus a dynamic process of continual

construction, rather than an accumulation of ready-made structures (von Glasersfeld, 1991, 1995).

Reacting to this very individualistic mechanism for learning, socioconstructivists looked for a stronger role for social interaction (Bauersfeld, 1992; cf. Lerman, 1996b). If we all construct our own reality, how is successful communication possible? “[H]ow is it that the teacher and the children manage to achieve at least temporary states of intersubjectivity² when they talk about mathematics?” Cobb, Wood, *et al* (1991: 162) asked, if all knowledge is acquired on an individual basis in response to individual needs. To the active construction of knowledge by the individual, Ernest (1991) added an essential role for experience and interaction in the physical and social worlds. This led to a third principle being added to the two quoted above from von Glasersfeld: reality is constructed intersubjectively (that is between individuals, as well as within them) and socially negotiated with significant others, with language playing a central role (Jaworski, 1994). Socioconstructivism focuses on the construction of human knowledge in and through communication, but it is still a constructivist paradigm (Lerman, 1996a). The basic premise is Piagetian, that knowledge is constructed by the individual first, but social interaction is seen as much more significant than in the radical constructivist position. In the socioconstructivist epistemology, language is central to a child’s development, providing tools of thought and carrying the cultural inheritance of that child’s community. Resolution of conflict and agreement on meaning are rooted in social interaction, but cognitive conflict is still seen as the prime cause of learning (Cobb, Wood and Yackel,

² ‘Intersubjectivity’ was the word used by socioconstructivists during the early and mid 1990s, to indicate understanding between individuals, not just within individuals.

1990). Despite the general individuality of constructivist epistemologies, in the socioconstructivist paradigm the child is obliged to construct knowledge which 'fits' with that of the classroom consensus (Bauersfeld, 1988). This also has implications for the teacher: s/he may no longer be the sole arbiter and controller of meaning, but may also need to modify her/his own knowledge as a result of discussion (Cobb, 1997, 1992; Taylor, 1996).

2.4 THE IMPACT OF VYGOTSKY

Vygotsky's work forms the foundation on which the theoretical positions taken in this thesis depend. This section contains a discussion of the circumstances in which he worked and his major theories. It starts with a brief overview of his life and work.

2.4.1 L. S. Vygotsky (1896 - 1934)

Vygotsky came to public notice with a lecture in 1924³. According to his future colleague, Luria⁴, this had an "electrifying effect" (Wertsch, 1988: 82) on his audience, not only changing the direction of Vygotsky's own life, but also that of Soviet psychology. Luria claimed that this lecture made Vygotsky "an intellectual force who would have to be listened to" (cited in Wertsch, 1988: 82), despite his lack of formal training in psychology. Vygotsky felt that none of the current frameworks would do as currently constituted, and that a new way of looking at psychological phenomena was needed, which would provide

³ *The Methodology of Reflexological and Psychological Studies*, delivered on 6 January 1924 at the Second Psychoneurological Congress in Leningrad.

⁴ A.R. Luria (1902-1977).

a “unified theory of human psychological processes” (Vygotsky, 1978: 5). He wanted to integrate a new theory of learning with a practical contribution to Soviet society, whose goal at the time was to develop the ‘new Soviet man’ (Graham, 1993; Wertsch, 1988).

However, Vygotsky never considered his conceptual scheme complete (Wertsch, 1985).

Between 1924 and his death in 1934, he attempted to open up new lines of thought which would explain how “higher mental functions”⁵ came into being, how they were related to each other, and the cultural context in which these events occurred (Vygotsky, 1978: 6).

He did not have the time before his early death to complete his theoretical framework, or to do the experimental work needed to support his ideas, but left this to his students and followers.

2.4.2 Background: the early twentieth century

Vygotsky and Piaget were both born in 1896, and both began their work in psychology against the background of the so-called “crisis in psychology” (van der Veer and Valsiner, 1991: 141) of the 1920s in Europe. Psychology was still a fairly new discipline, and it was not yet clear whether it was to be a natural science or the “science of the soul” (p 151).

Vygotsky, like many others at the time, thought that psychology lacked a clear theoretical basis to support its concepts and explanatory principles, and that it needed a methodology. He believed that in the post-revolutionary USSR there was a unique opportunity to resolve this crisis, since society was starting anew from a “clean slate” (Rieber and Wollock, 1997b: viii).

⁵ These included “voluntary attention, voluntary memory and rational volitional, goal-directed thought” (Vygotsky, 1987: 20).

Vygotsky knew of the work of the behaviourists of his time, and of Piaget's early work. He had a considerable respect for Piaget, writing: "[p]sychology owes a great deal to Jean Piaget." (Vygotsky, 1986: 12). In Chapter 2 of *Thinking and Speech* (1987, written originally in 1932) Vygotsky gave a detailed critique of Piaget's work. While agreeing with many of Piaget's ideas, especially his experimental method and interpretation, he disagreed with Piaget's developmental stage theory⁶, claiming that learning and development have a complex interrelationship, and that development does not lead to learning, as Piaget claimed. He also felt that what was missing in Piaget's work was "reality", and the child's relationship with "reality", by which he meant the child's practical activity (Vygotsky, 1987: 87).

In addition to the fragmented state of psychology in Europe at the time, another important factor in the development of Vygotsky's thought was the Russian revolution of October, 1917. The years 1917-1932 saw a major transformation in Soviet science. The immediate post-revolutionary period was one of great ferment, with many new ideas in the air, and intellectual control not yet the major hindrance it later became. Many areas were opened up in science which were applied to building up the new state (Wertsch, 1985). By the mid 1920s, scientists wanted to build a new Marxist psychology, starting from the practical activity of human beings (Leont'ev in Rieber and Wollock, 1997a). With the rise of Stalin, however, ideological imperatives became much more important, and there was a growing suspicion of Western science resulting in the increasing isolation of Soviet science.

⁶ Piaget claimed that all children develop through a succession of stages. The age at which they reached a new stage could vary, but the progression was invariant. It would not be possible for a child to develop in a way characteristic of a later stage than the stage they had reached.

2.4.3 Marxist basis for Vygotsky's work

Vygotsky saw dialectical materialism⁷ as a coherent basis on which a new psychology could be developed, agreeing that:

... the Marxist dialectic ... is a general orientation and culture of thought that helps each person to pose a problem with greater clarity and purpose and thereby helps him to solve the riddles of nature. (Semenov, 1968, cited in Graham, 1987)

He further believed that “the methods and principles of dialectical materialism [provided] a solution to key scientific paradoxes” (Vygotsky, 1978: 6, *Introduction*), seeing “child development [as] a complex dialectical process” (Vygotsky, 1978: 73). Vygotsky’s views on the mediation of tools and the influence of the social environment are also characteristic of dialectical materialism; indeed, he used Marx’s writings as a source in developing his ideas:

Marx said long ago that “the use and creation of implements of labor, although present in embryonic form in some species of animals, are a specific characteristic of the human process of labor” (Marx, 1920, p153). (Vygotsky, 1986: 90)

However, many of Vygotsky's references to Marxism were dropped in the first translations of his work into English (Vygotsky, 1934/1962), the translators and editors seeing these as unimportant rather than an essential foundation.

Vygotsky began his work in psychology in 1924 by dissenting from Kornilov’s approach. Kornilov was then Director of the Institute of Psychology in Moscow, to which Vygotsky had been invited in 1924. He was studying reactology, which was the interaction between organisms and their environments, linking it to Marxism (van der Veer and Valsiner,

⁷ Dialectical materialism was a philosophical theory of science, deriving from Marx’s, Engel’s and Lenin’s writings. Materialism has at its roots the desire to seek a rational, scientific explanation for everything, and to explain all of nature in terms of matter and energy. In a dialectical approach, all phenomena are seen to be in a process of perpetual change. In Soviet Marxism, man and nature were seen as one, and any attempt to explain either had repercussions for the other. (e.g. Appighanesi, 1976; Graham, 1987, 1993).

1991). Vygotsky felt that Kornilov and others like him were using quotes from Marxist literature, but were not really using Marxist theory as a true foundation for their work. Vygotsky's theoretical basis was to use Marxism as a scientific method, using the methods and principles of dialectical materialism (Vygotsky, 1978: 60f; Wertsch, 1996). All psychological functions were to be studied as processes in motion and change rather than as "stable, fixed objects" (Vygotsky, 1978: 61). This led him to "developmental psychology, not experimental psychology, [as this] provides the new approach to analysis that we need" (p61). This emphasis on studying the history and development of human phenomena was characteristic of Vygotsky's work throughout his life, and it derives directly from Marxist theory.

2.4.4 Fundamental themes of Vygotsky's thought

The basis for Vygotsky's theory was the Marxist view that to understand higher mental processes in individuals, you first have to understand their social context. He developed three fundamental themes (Wertsch, 1988). The first was his genetic or developmental method. By this Vygotsky meant that higher mental functions should be studied through their history: "[i]t follows, then, that we need to concentrate not on the *product* of development but on the very *process* by which higher forms are established." (Vygotsky, 1978: 64). His second theme was that "[a]ll the higher functions originate as actual relations between human individuals" (p57), emphasising the social origin of higher mental functioning. The third theme was that such higher mental functioning is mediated by socioculturally evolved tools and signs, of which human language is the most significant. It is this third theme which is most important in building up the theoretical framework on which this thesis is based.

Mediation by tools and signs

According to Engels, using tools is fundamental to the transformation of apes into humans:

The specialisation of the hand – this implies the *tool*, and the tool implies specific human activity, the transforming reaction of man on nature... (Engels, 1934: 34)

This became part of Marxist doctrine. Vygotsky:

... was the first to attempt to relate it to concrete psychological questions. ... he creatively elaborated on Engels' concept of human labor and tool use as the means by which man changes nature and, in so doing, transforms himself ... (*Introduction*, Vygotsky, 1978: 7)

However, where Engels viewed tools as the means by which humans interact with their environment, Vygotsky extended this to encompass sign systems also. To him, tools and signs were the same in their mediating function. The difference is that we use tools to act on objects, whereas we use signs, or psychological tools, such as "language, writing, number systems" (Vygotsky, 1978: 7), to change our own behaviour (p54f). Tools and signs are therefore crucial in determining the nature of learning.

If one changes the tools of thinking available to a child, his mind will have a radically different structure. (Berg, 1970: 46, cited in *Afterword*, Vygotsky, 1978: 126)

Unlike Piaget, Vygotsky's basis for understanding human development was not the maturation of the individual through adaptation to the environment, but human activity and the use of tools and signs. Tools and signs are developed by people working together and have to be mastered by a child through social interaction (van der Veer and Valsiner, 1991) and through the use of language (Vygotsky, 1978: 23). Using his own theoretical explanations together with experimental work carried out by his colleague R.E. Levina, he concluded:

... the most significant moment in the course of intellectual development, which gives birth to the purely human forms of practical and abstract intelligence, occurs when speech and practical activity, two previously completely independent lines of development, converge. (Vygotsky, 1978: 24, original italics)

This conjunction between practical activity using tools and the use of language as a problem-solving symbolic tool is fundamental to the case made in this thesis for the use of the graphic calculator as a mediating tool.

The zone of proximal development

In assessing the developmental level a child has reached, Vygotsky felt it was important to look not only at what a child could do alone, but also to look at what the child could do if helped. Vygotsky gave the example of two eight-year-olds. One could perform at the twelve-year old level when assisted, whereas the other was only able to perform at the nine-year old level (Vygotsky, 1978: 85f). Clearly, the first child is at a different level of development from the second. Vygotsky called the difference between the child's performance alone and when aided the 'zone of proximal development' (ZPD):

It [the zone of proximal development] is the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers. (Vygotsky, 1978: 86, original italics)

What lies in the zone of proximal development at one stage is realized and moves to the level of actual development at a second. In other words, what the child is able to do in collaboration today he will be able to do independently tomorrow. (Vygotsky, 1987: 211)

It is a contention of this thesis that two children working together with a graphic calculator can form a ZPD, enabling both children to advance further than they would have done alone or unaided by the graphic calculator.

Scientific and everyday concepts

Piaget succeeds in differentiating spontaneous and nonspontaneous concepts, but does not see that they are united in a single system that is formed in the course of the child's mental development. He sees only the break, not the connection. (Vygotsky, 1987: 174)

Whereas Piaget believed that children started with spontaneous concepts, which were later superseded, Vygotsky believed these were part of a single system. ‘Scientific’⁸ concepts were those that the child acquired during formal instruction⁹, ‘everyday’ concepts were those met during everyday life. Vygotsky hypothesised that scientific concepts move downwards to meet corresponding everyday concepts, and that stable concept formation occurred when they interacted:

The development of scientific concepts *begins with the verbal definition*. As part of an organized system, this verbal definition descends to the concrete; it descends to the phenomena which the concept represents. In contrast, the everyday concept tends to develop outside any definite system; it tends to move upwards toward abstraction and generalization. (Vygotsky, 1987: 168, original italics)

This hypothesis led Vygotsky to claim that learning concepts through direct instruction alone is impossible, leading to the “mindless learning of words, an empty verbalism that simulates or imitates the learning of concepts in the child” (p170). Teaching a new concept is not the end of the process of internalising it, but the beginning (p172).

This interaction of practical, everyday ideas and the more formal, abstract ideas taught in school is utilised in the graphic calculator model described in this thesis. Connections between everyday practical activity and ideas and the more abstract ideas of algebra are facilitated by the graphic calculator model of a variable, (discussed later in this chapter in section 2.7.2) and by the classroom modules in which it was used (section 4.4).

⁸ “In using the term ‘scientific’ in this context, Vygotsky is emphasising (1) the *systematic* nature of scientific knowledge and (2) its association with the peculiar social institutions of science and education.” (*Editor’s Notes*, Vygotsky, 1987: 388)

⁹ “The term that is translated here as ‘instruction’ (*obuchenie*) has been translated in other texts as ‘learning’. Neither of these English glosses is an entirely adequate translation of the Russian term. ... Thus the term *obuchenie* seems to us to imply the teaching/learning process involved in instruction; not merely the action of the instructor or the learner.” (*Editor’s Notes*, Vygotsky, 1987: 388)

Language and discussion

The connection between discourse and concept forms an essential aspect of Vygotsky's theory:

Learning to direct one's own mental processes with the aid of words or signs is an integral part of the process of concept formation. (Vygotsky, 1986: 107)

Once a concept is explicated in dialogue, the learner is enabled to reflect on the dialogue, to use its distinctions and connections to reformulate his own thought. (Bruner, 1987: 4, *Prologue* to Vygotsky, 1987)

Piaget and Vygotsky agreed that the child's thinking:

... must ... be understood, and perhaps primarily so, as a function of those relationships which are established between the child and the social environment that surrounds him. ... The very structure of the individual's thinking depends on the social environment. (*Preface* to the Russian edition of *Judgement and Reasoning in the Child*, Piaget, 1928, cited in Vygotsky, 1987: 82)

However, Vygotsky, unlike Piaget, did not believe that concept development is essentially biological. Where Piaget theorised concept development as a process of adaptation between the individual and the environment (Piaget, 1974/1977), Vygotsky saw it as a process in which social phenomena are transformed into psychological phenomena through the mediation of signs (that is, symbolic tools, particularly language) and tools (Vygotsky, 1978). Vygotsky felt that Piaget radically underestimated the significance of language, believing that:

Real concepts are impossible without words, and thinking in concepts does not exist beyond verbal thinking. That is why the central moment in concept formation, and its generative cause, is a specific use of words as functional "tools". (Vygotsky, 1986: 107)

Throughout his career, Vygotsky consistently emphasised the importance of language in mediating psychological processes (Minick, 1996).

Another part of the theorisation of the graphic calculator provided in this thesis is that the graphic calculator, in providing mathematical objects on which to operate, also provides

objects for discussion and appropriate language for that discussion, thus assisting the formation of concepts.

The role of the peer/teacher

Vygotsky believed that adult guidance and collaboration with peers were as necessary to learning as the exposure of the student to new material (Vygotsky, 1978: 86; 1987: 187). The teacher or peer does not always need to be physically present, as the memory of discussion can often be sufficient to enable the learner to move forward (Daniels, 2001; Vygotsky, 1987: 216). Peer collaboration was also seen as an effective medium for learning by Piaget and his colleagues (Brown, *et al.*, 1996a; Phelps and Damon, 1991), enabling the learners to work at a level higher than they would alone (Brown, Campione, Ferrara, Reeve and Sullivan Palincsar, 1991; Landsmann, 1991; Richards, 1991). This is also an important strand in the theorisation of the graphic calculator provided here.

2.4.5 Socioculturalism

Socioculturalists, who are Vygotsky's successors, have criticised the behaviourists of the early and mid-twentieth century and all types of Piagetians and constructivists (including socioconstructivists) for locating learning in the individual (Brown, Stein and Forman, 1996b; Lerman, 1996a):

The metaphor of students as passive recipients of a body of knowledge is terribly limited: so too is the metaphor of students as all-powerful constructors of their own knowledge, and indeed of their own identities. (Lerman, 1998: 70)

Like Vygotsky, socioculturalists see learning as occurring first in the social plane rather than in the individual. This is fundamentally opposed to the Piagetian view, despite

Piaget's view of the importance of the social environment, since for Piaget learning occurred primarily in the individual.

Socioculturalists are critical of socioconstructivists. For instance, Lerman saw the socioconstructivists' position as essentially constructivist to which a layer of social interaction has been added (Lerman, 2003, cf. Cobb, *et al.*, 1990). He drew attention to the danger of bolting together theories which at heart are contradictory and which do not address each other's weaknesses. Zevenbergen saw further dangers, highlighting the risk of individualistic theories of learning legitimising the marginalisation of many social and cultural groups (1996):

Within a constructivist paradigm, the individual construction of meaning recognises that students from social, cultural and gendered groups are likely to construct different meaning based on their past experiences and it is this which can be seen to be responsible for the poor performance of many marginal social groups. (p105)

Currently socioculturalists are engaged in developing Vygotsky's theories for the cultures of our time. Daniels (1996) pointed out that the Vygotsky of the USSR in the 1920s and 1930s is not the Vygotsky of the West post-1970; the Vygotsky of the West of the late twentieth and early twenty-first centuries must also be seen in a cultural context.

Nevertheless Wertsch (1988) has claimed that Vygotsky's work has provided an "overarching theoretical framework" (p88) needed in Western thought, together with a fundamental shift from the individual to the collective (cf. Daniels, 2001; Minick, 1996; Wertsch and Tulviste, 1996). However, Wertsch also pointed to the dangers of distortion and the use of Vygotsky's theories piecemeal (cf. Lerman, 1996a).

Development of Vygotsky's work in this way is very necessary. Vygotsky did not have the time to develop his theories fully. He was aware of his impending death, so initiated new

theory, rather than working on establishing it, leaving this to those who came after him. At his death in 1934, there was a considerable amount of further work to be done, some of which was done by his colleagues in the USSR (led by Leont'ev of the Kharkov group¹⁰). More recently, with the opening up of the USSR and the emigration of Russian scholars to the West, development of Vygotsky's work has proceeded in the West also. The Kharkov group concentrated on activity theory¹¹ (e.g. Bakhurst, 1996; Kozulin, 1996; Minick, 1996), which was more overtly Marxist than Vygotsky's primacy of language and culture, and hence more acceptable to Stalin. In the West, scholars have considered tool mediated action (e.g. Bakhurst, 1996; Daniels, 2001) or goal directed action (Daniels, 1996) as a starting point. There has also been considerable expansion of Vygotsky's concept of the zone of proximal development (e.g. Brown, *et al.*, 1996a; Daniels, 2001; Lave and Wenger, 1996; Wertsch and Tulviste, 1996). Engeström (1996) has discussed further developments of Vygotsky's theory of concept development through the meeting of everyday and scientific concepts (cf. Hedegaard, 1996).

2.5 MEDIATION OF TOOLS IN THE LEARNING PROCESS

There is nothing new about using tools to help students learn mathematics.

The development of mathematics has always been dependent upon the material and symbolic tools available for mathematical computations. Nobody would deny the role played by the introduction of the decimal system, the construction of logarithmic

¹⁰ Shortly before Vygotsky's death, growing ideological pressure caused a group of psychologists, including Vygotsky's colleagues, Leont'ev and Luria, to move from Moscow to Kharkov. Vygotsky was also invited to join them, but preferred to remain in Moscow (van der Veer and Valsiner, 1991: 185).

¹¹ Activity theory has been "the chief category of psychological research in contemporary Soviet psychology since the beginning". It originated in the work of Vygotsky, who suggested that "socially meaningful activity may serve as an explanatory principle in regard to, and be considered as a generator of, human consciousness." However, in the mid 1930s, "a group of Vygotsky's disciples came up with a 'revisionist' version of activity theory that put practical (material) actions at the forefront while simultaneously playing down the role of signs as mediators of human activity." (Kozulin, 1996: 99)

tables, the tabulation of elementary functions, or the development of mechanical and graphical computational tools. (Artigue, 2002: 1)¹²

Theorisation of the use of such tools has developed mainly since Vygotsky's seminal work on this subject, becoming particularly significant since the development of the computer and computerised technologies in general. However,

What is firstly asked of software and computational tools is to be pedagogical instruments for the learning of mathematical knowledge and values which were defined in the past, mostly before these tools existed. (Artigue, 2002: 2)

The mathematics that is currently taught to students relates to an era before computers were readily available. There is still suspicion of methods which cannot be replicated using paper and pencil technology only: for example, students taking A level examinations in the UK are still required to show that they can answer questions without any calculating aids¹³. Yet no practising mathematician would handicap themselves by working without appropriate professional tools. That it does hamper students to work without the tools they have become accustomed to is made clear by Noss and Hoyles:

... these tools wrap up some of the mathematical ontology of the environment and form part of the web of ideas and actions embedded in it ... (1996: 227).

It is apparent that there is still a need for a theorisation of tools which provides an explanatory structure for their role in mathematics education. In the specific case of the graphic calculator, since it is more like a hand-held computer than a simple calculator (Penglase and Arnold, 1996; Ruthven, 1990), much that has been said and written about the use of computers also applies to them.

¹² These page numbers refer to a pre-publication copy of Artigue's paper.

¹³ Information obtained from http://www.qca.org.uk/subjects/downloads/revised_gce_maths_criteria_20040105.pdf on 7 April 2004.

2.5.1 The computer as a technology that transcends the limitations of the mind

In 1985, Pea defined a *cognitive technology* as

... any medium that helps transcend the limitations of the mind, such as memory, in activities of thinking, learning, and problem solving ... (p168).

It was axiomatic, he claimed, citing Vygotsky (1934/1962; 1978) in support of his argument, that:

... intelligence is not a quality of the mind alone, but a product of the relation between mental structures and the tools of the intellect provided by the culture ... (Pea, 1985: 168)

Technologies which have been most explored in this context include “written language ... and systems of mathematical notation, such as algebra or calculus”. Pea then went on to assert that “computers may provide the most extraordinary cognitive technologies ... devised” (p168).

Computer technology has had many other benefits claimed for it over the last decade or two. It could aid students in exploring mathematical relationships (Dreyfus, 1993; Hoyles, 1993; Noss, 1998) or in reflective abstraction (Ayers, Davis, Dubinsky and Lewin, 1988; Dörfler, 1993; Hoyles and Noss, 1992). Computers can take over the technical parts of an investigation or calculation, leaving students free to concentrate on the conceptual aspects of a problem (Dörfler, 1993; Dreyfus, 1993; Pea, 1985, 1986, 1987). Computers can also change the nature of the learning process or the nature of what is learnt (Dörfler, 1993; Papert, 1980; Pea, 1985; Shaffer and Kaput, 1999). They enable learning to be provisional: hypotheses can be made, tested, modified, thrown out and replaced easily; mistakes can be removed without trace (Fox, Montague-Smith and Wilkes, 2000; Ruthven and Hennessy, 2002). Others point out that learning rarely occurs in a linear fashion, and that undue simplification can cause problems later (Choi and Hannafin, 1995), whereas use

of the computer allows more complex ideas and real data to be introduced much earlier in the learning process (Tall, 1989).

However, while agreeing with much of the above, some researchers are more cautious about the effects of computer technology on learning. Ruthven (1993) observed a lot of trial and error which was unsupported by reflection; Keitel, Kotzmann, *et al* (1993) claimed similarly that a 'black box' is substituted for mathematical processes, and the mathematics is never made explicit. Fox, Montague-Smith, *et al* (2000) asked whether students learn from their errors when working at the computer, or whether they just become more confused.

2.5.2 Metaphors for tool action

Some tools can be classified as "defining technologies": these redefine man's role in relation to nature, such as the potter's wheel, the clock, the steam engine, and the computer (Bolter, 1984: 11; Salomon, 1991: 186f). Defining technologies enable people to perform new tasks, or to ask new questions and find new answers, or to make new distinctions.

According to Salomon, such defining technologies are metaphors which serve as "cognitive prisms" (p186f), through which we can examine and interpret other phenomena.

An example of a technology which acts in this way is the clock:

The situation is much like that of making a clock and letting it run and continue its motion by itself. In this manner God allows the heavens to move continually ... according to the established order ... (Oresme, 14th century, cited in Salomon, 1991: 187)

Here Oresme is using the clock as a metaphor to say something about the nature of heaven.

The clock is a "defining technology", which "develops links, metaphorical or otherwise, to a culture's science, philosophy, or literature; it is always available to serve as a metaphor, example, model, or symbol" (Bolter, 1984: 11). A defining technology is one that helps us

to make sense of the incomprehensible, in this case the heavens. These metaphors also help us to organise our knowledge and to guide our exploration of new phenomena. An example of a metaphor creating such a guide is the comparison of the mind to a computer. However, although such metaphors can draw our attention to certain aspects of, for instance, the heavens or the mind, that direction may be at the expense of other facets (Salomon, 1991).

Another metaphor in common use to exemplify how the computer acts as a tool for learning is that of 'amplification'.

Computers are commonly believed to change how effectively we do traditional tasks, amplifying or extending our capabilities, with the assumption that these tasks stay fundamentally the same. (Pea, 1985: 168)

Pea (1987) believed, however, that amplification is not an adequate metaphor to describe the effect of cognitive tools or technology. Amplification describes the process whereby the computer extends our ability to perform a task, but does not change the nature of the task, or the way that the task is viewed. Pea claimed that tasks are changed by cognitive technologies, changing also the way they are conceptualised. To describe this, he used the metaphor of "reorganization" (1985: 170f). The effect of cognitive tools is to not only extend or speed up some human faculty, but also to change the way we think: they are "reorganizers of mental functioning" (p179).

These ideas are in many ways implicit in the mediation of signs in Vygotsky's framework and that of the socioculturalists. Signs cause changes in a person, who then goes on to change the sign, in a continuous dialectical process (Pea, 1985; Vygotsky, 1978: 54ff; Wertsch, 1985).

These signs are special psychological tools by means of which the individual organizes his behavior and learns to direct them voluntarily. Just like tools of labor, they act as an intermediate link between the activity of the person and the external

object and mediate the relationships between them. But whereas the tools of labor are directed toward the object and change it according to a consciously set goal, the signs change nothing in the object, but serve as a means by which the subject can influence himself, his own mind. (Yaroshevsky and Gurgendze, 1997: 350, *Epilogue to Vygotsky, Collected Works, Vol III*)

2.5.3 Instrumentation of tools

Fears mentioned at the end of section 2.5.1, that computers may act simply as ‘black boxes’ with no gain in conceptualisation by the user, have caused a group of French researchers to rethink the commonplace view that the use of technology will necessarily support learning. If such cognitive technologies are to be used to assist students of mathematics, the question of how the tool can become a means to access the mathematics needs to be considered (Artigue, 2002).

The widespread idea that computer environments, because they can appear to take on technical aspects of mathematical activity, spontaneously induce mathematical activity, which is both more reflective and conceptual, must be challenged. (Guin and Trouche, 1999: 200)

Guin and Trouche refer to this process, whereby the mathematics implicit in the use of the tool becomes explicit, as ‘instrumental genesis’ or ‘instrumentation’:

Transforming any tool into a mathematical instrument for students involves a complex ‘instrumentation’ process and does not necessarily lead to better mathematical understanding. (Guin and Trouche, 1999: 195)

The process of instrumentation (that is, of transforming an artefact into a transparent mathematical tool) takes time, and needs to be carefully considered (Artigue, 2002; Guin and Trouche, 1999; Lagrange, 1999; Vérillon and Rabardel, 1995).

Initially, a tool is merely an ‘artefact’: this is an object whose purpose the user is not aware of, so it lacks the potential to help the user in the way intended. For instance, a calculator used as a straight edge is simply an artefact (Monaghan, 2003). If the artefact is to become an instrument (the calculator becomes a calculating aid rather than a straight edge), the

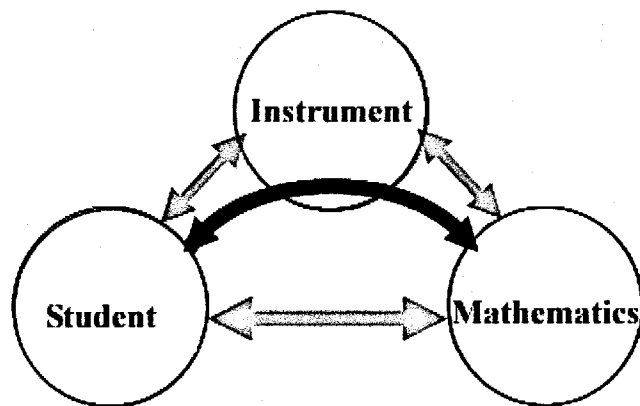
user has to enter into a relationship with it, which allows it to become an agent for doing mathematics. The user needs to appropriate the tool for themselves and integrate it with their activity (Monaghan, 2003; Vérillon and Rabardel, 1995). Studies conducted by the French group:

... clearly show that the complexity of instrumental genesis has been widely underestimated in research and innovation ... until quite recently. (Artigue, 2002: 8).

Artigue attributed the lack of attention to instrumental genesis to the predominant role given to technology as a pedagogical tool:

Suggesting that instrumentation may be a complex and costly process does not fit visions that consider technology mainly as an easy tool for introducing students to mathematical contents and norms defined independently from it. (p8)

The diagram below (Figure 3) is adapted from Ruthven (2003), which is based on a diagram in Vérillon and Rabardel (1995: 85). It illustrates the conceptualisation of an artefact becoming an instrument through its relationship with the student and the mathematics. Although there are direct relationships between each of these (shown by the green arrows), there is also a relationship between the student and the mathematics, which is mediated by the instrument (shown by the black arrow). The process of instrumentation is the establishment of this relationship, and it is the creation of this mediating relationship that changes an artefact into an instrument or tool.

Figure 3: Process of instrumentation

It is in this way that the use of the graphic calculator by students is conceptualised in this thesis. In this theorisation, the calculator becomes an instrument which is transparent (Ruthven, 2003): the tool does not hide the mathematics, but reveals it. The examples given in Chapter 5 exemplify this, with the students developing their conceptual understanding through the mediation of the graphic calculator.

2.5.4 The graphic calculator as a mediating tool

This thesis is about the use of the graphic calculator as a mediating tool. The graphic calculator is a form of cognitive technology (see section 2.5.1), which can enhance the capability of a student. It also has the potential to be a ‘cognitive prism’ providing a metaphor to explain the incomprehensible (see section 2.5.2). As is discussed later (section 2.7.2), the calculator provides a model of a variable which helps students to access a difficult concept early in their experience of algebra. My initial expectation of how it would help students was as an ‘amplifier’, performing routine tasks, and thus giving students time to access deeper conceptual understanding. This expectation is also found in Berger (1998). In fact, this was not the most significant effect that I found, although it was

for Berger. The evidence of cognitive change was much more significant, as discussed in Chapter 5 (cf. Gage, 2002b), which accords with Pea's (1985) claims.

There is little research that looks specifically at how students working with the graphic calculator access knowledge. The literature on graphic calculator research so far is mainly about specific ways of using the calculator to aid students in learning a given aspect of the curriculum, particularly graphs. There is much less research which looks at how it does this, or what impact it might have on teaching and learning. In 1996, after a decade of graphic calculator use in schools, Penglase and Arnold asked: "In what ways can [graphic calculators] be used to maximise learning and achievement?" (p59). They concluded that an answer to this question remained "elusive and conflicting", and that the current state of research was inconclusive.

A year later, Hennessy (1997) reviewed the literature on portable technologies in a report for the Computers and Learning Group at the Open University. She found studies which claimed that handheld technologies were empowering, assisting student autonomy and active learning. Key features which contributed to the usefulness of handheld technologies included accessibility, portability, ease of use, and time for experimentation. The graphic calculator scores highly on all these indicators, as Hennessy recognised. Both Penglase and Arnold and Hennessy focused mainly on how the graphic calculator supports students in the area of graphing.

In 2002, Burrill, Allison, *et al* conducted a review of graphic calculator research in a variety of different mathematical areas. They found that there were still many unanswered questions about how the technology is used, what its impact on student understanding is,

and which students benefit from using the technology. They also found few studies on use by children less than 16 years of age. They concluded that research so far is 'uneven', but that it at least establishes the groundwork for more rigorous research in the future.

It is clear from these studies that the graphic calculator has the potential to be an effective classroom tool for a range of reasons. It can aid the active construction of learning by giving immediate feedback (Gage, 2002b; Hennessy, Fung and Scanlon, 2001) and reducing drudgery (Hennessy, 1997). It can also lead to collaborative work (Hennessy, *et al.*, 2001), focused dialogue and reflection (Graham, 1998) which all aid the constructive process (Gage, 2002a). Ruthven (1995) observed informal networking when year 7 students used graphic calculators as part of their normal classroom kit. The students that he observed felt that it was a useful learning tool, and it was found to have an overall positive influence on their attitudes to the use of technology. It provides a private space for an individual to try out hypotheses (Penglase and Arnold, 1996) without mistakes being made public (Hennessy, 1997), and it has all the provisionality of computer technology (Gage, 1999a; Ruthven and Hennessy, 2002). Like other forms of computer technology, the graphic calculator can externalise objects for reflection, analysis and discussion, providing a physical reality for abstract concepts (Graham, 1998; Graham and Thomas, 2000a).

In *Mind and Society* (1978: 40-45), Vygotsky described a series of experiments in which pre-school children (aged 5-6 years), older children (aged 8-9 years), adolescents (aged between 10 and 13 years) and adults were all given external aids to help them in a memory-based task. The very young children could not make use of the aids, and found them confusing (this corresponds to artefacts which have not become tools through the

process of instrumentation, as described in the previous section). The older children and the adolescents found the external aids very useful, but the adults also found them confusing, preferring to use their own internal resources. Vygotsky found that the youngest children could not yet make use of external prompts to mediate their mental functioning, school children and adolescents found them very useful, but the adults had already internalised such signs, and no longer needed them:

... the external sign that school children require has been transformed into an internal sign produced by the adult as a means of remembering. This series of tasks applied to people of different ages shows how the external forms of mediated behavior develop. (Vygotsky, 1978: 45)

This result suggested that for children in the age group I was studying (10-14 years), using an external instrument to help them in their learning might indeed prove valuable.

2.6 PERCEIVED DIFFICULTIES IN THE TEACHING AND LEARNING OF ALGEBRA

The previous sections of this chapter have considered the epistemology underlying the work described in this thesis, and the work of Vygotsky on which it is grounded.

Mathematics is seen as a human construct, which has to be actively built up by each individual. The tools that mediate this construction are an important consideration when discussing how students form concepts. In the context of the research described in this thesis, the graphic calculator acts as a form of cognitive technology, which can act as a mediating tool in the way that Vygotsky described. The next sections of this chapter consider aspects of the teaching and learning of algebra, and how the graphic calculator can contribute to these.

2.6.1 The nature of algebra and thinking algebraically

What is algebra and what does it mean to think algebraically? What is the connection between algebra and arithmetic? Most mathematicians, mathematics educators and researchers find it easy to recognise algebra and algebraic thought, yet equally most find it very difficult to characterise them (e.g. Charbonneau, 1996; Janvier, 1996; Lins, 1992; Usiskin, 1988).

Algebraic statements usually involve a specific notation, yet working with letters or strings of symbols may not necessarily be evidence of algebraic thinking (Janvier, 1996; Love, 1986; Mason, 1996). As Wheeler (1996a) pointed out, algebra certainly involves working with a symbolic system, but it is much more than that. $3(x + 5) + 1$ may be the result of an algebraic process; it might equally be an empty string of symbols (Sfard and Linchevski, 1994). Janvier suggested that using the formula πr^2 to calculate the area of a circle from its radius does not involve algebraic thinking, whereas using this formula to calculate the radius of a circle from the area does.

The purpose of using the graphic calculator in the classroom studies described later in this thesis (section 4.4.3) was to give students a way to find meaning in such an expression, and a context for working with it, so that it became more than an empty string of symbols for them. It was hoped that this grounding would enable the students to move forward into genuine algebraic thinking:

Algebra is now not merely 'giving meaning to symbols' but another level beyond that: concerning itself with those modes of thought that are essentially algebraic – for example, handling the as-yet-unknown, inverting and reversing operations, seeing the general in the particular. Becoming aware of these processes, and in control of them, is what it is to think algebraically. (Love, 1986)

2.6.2 Comparisons between the historical development of algebra and difficulties students experience in learning algebra today

Many writers have looked at the historical development of algebra (phylogenesis) and drawn parallels between this and the way that individual children learn (ontogenesis), although they have not always agreed about the stages of that development. Harper (1987) identified the main stages of algebraic development as rhetorical (where the problem and solution are framed in natural language, as was the case before Diophantus), syncopated (where a mix of words and symbols was used, as happened from Diophantus¹⁴ until the end of the 16th century, particularly in the work of Viète¹⁵ in the mid-16th century) and symbolic. Sfard (1995) gave the main stages as rhetorical and syncopated (pre-Viète), Viètan symbolic algebra, and abstract algebra, which she claimed corresponded to primary, secondary and tertiary stages of education.

The development from rhetorical to syncopated to symbolic forms of algebra can certainly be used to give a structure to model the development children have to go through in their learning, if they are to be successful users of algebra (Sfard and Linchevski, 1994). Harper (1987) suggested that the curriculum should mirror the history of mathematics, and looked at this in detail for the case of algebra. He concluded that using the methods of history to introduce algebra could help students' attempts to find some meaning in the tasks they

¹⁴ Diophantus of Alexandria (his dates are uncertain, but he is generally assumed to have lived around 250AD) was a leading algebraist of the Greek period. He used words and some abbreviations in his algebraic methods, including a symbol for an unknown. This is the type of notation known as 'syncopated' (Boyer, 1968: 197ff)

¹⁵ François Viète, or Franciscus Vieta, as he was also known, (1540-1603), developed algebraic notation, distinguishing between a parameter and an unknown. Nevertheless, his notation was still essentially syncopated rather than symbolic (Boyer, 1968: 333ff)

were expected to do, but that it was not realistic to expect some 1300 years of history to happen in the classroom in a mere five years. While this paralleling of stages in the development of algebra as a discipline and that of its development in the individual child has many advocates, others believe that the lessons of history are less direct than this (Wheeler, 1996b).

2.6.3 Difficulties at the arithmetic/algebra interface

Some definitions of algebra equate it to a form of generalised arithmetic:

Historically, algebra grew out of arithmetic and it ought so to grow afresh for each individual. (Mathematical Association, *The Teaching of Algebra in Schools*, 1929/1945: 5, cited in Lee and Wheeler, 1989: 41)

Yet for Love (1986) algebra is about handling the “as-yet-unknown” (p 49), which is not an arithmetic process (cf. Janvier, 1996). Usiskin (1988) claimed that algebra is not just generalised arithmetic, and it is more than a vehicle for problem-solving:

It provides the means by which to describe and analyze relationships. And it is the key to the characterization and understanding of mathematical structures. (p18)

Kieran (1988b: 91) claimed that “[a]lgebra is often called ‘generalized arithmetic’” and Demana and Leitzel (1988) felt that students need to work with key algebraic concepts in a numerical setting first, since the basic concepts of algebra are available through numerical experience. The many attempts that have been made to characterise the relationship between arithmetic and algebra suggest that this is as difficult as pinning down the nature of algebra itself.

Clearly there is an intimate connection between arithmetic and algebra. However, as Lee and Wheeler (1989) pointed out, the connection between arithmetic and algebra is not always obvious, particularly when students first meet algebra (cf. Herscovics and Linchevski, 1994; Matz, 1980). Nor is it obvious to them why they have to use algebraic

methods, when often informal or arithmetic methods work perfectly well (Lee and Wheeler, 1987).

Students' and their teachers' difficulties in moving from arithmetic to algebra have been well-documented (e.g. Dickson, 1989; Linchevski and Herscovics, 1996; Sfard and Linchevski, 1994; Usiskin, 1988). In arithmetic, signs mean 'do something', so, for instance, $3 + 5$ is interpreted as the operation of addition. In algebra these same signs are no longer instructions to do something, but are the expression of relationships: $a + b$ is a mathematical object in its own right, and the '+' sign expresses the relationship of the variables a and b , rather than an instruction (e.g. Booth, 1988; Kieran, 1990; Nickson, 2000). In arithmetic, students operate on numbers, in algebra they have to learn to operate on objects like $a + b$. This is often a major difficulty for them (Nickson, 2000).

Another cause of difficulty may arise from methods being masked in arithmetic. When children do arithmetic, they make specific calculations, which generally have a specific numerical answer (Booth, 1988). Achieving a correct answer may allow difficulties or deficiencies in methods to go unappreciated by either the child or the teacher, and these may then transfer over into algebra, where they cause greater problems (Booth, 1984; Kieran, 1990). Misconceptions in arithmetic may also be passed on. Booth gives the example of $12 \div 3$ and $3 \div 12$, which students often think mean the same thing, because you just divide the larger figure by the smaller (cf. Dickson, 1989).

It is very common for children to spend several years learning arithmetic before they start algebra, and their early use of algebra is rooted in the procedures and concepts of arithmetic (Bednarz and Janvier, 1996). However this can cause problems, with the child

experiencing disturbances to their arithmetic (e.g. Booth, 1984; Carraher, Schliemann and Brizuela, 2000; Filloy and Rojano, 1989; Sfard and Linchevski, 1994). Lee and Wheeler (1987; 1989) found that about a third of the students they interviewed (aged 15 or 16 years) believed that $20 = 4$ was an acceptable answer when algebra was involved, since these students had no expectation that arithmetic and algebra would obey the same rules:

As in the previous problems, students gave a justification by rule for the algebraic development. That these “rules” could lead to a result which is nonsense in arithmetic did not appear to be a problem for the majority of these students. ... Once again students behaved as though algebra were a closed system untroubled by arithmetic. (Lee and Wheeler, 1989: 45f)

This disassociation of arithmetic and algebra may mean that students do not check their algebraic answers numerically, not seeing this as a useful thing to do.

Ávalos (1996) however found that when 11-12 year-old children used graphic calculators to begin their study of algebra, many of the problems commonly experienced were avoided. The children in his study used their knowledge of arithmetic and language provided by the calculator to generalise about relationships between variables. He found that there was no disassociation between arithmetic and algebra, and that the arithmetic background provided meaning for the children because they encountered algebra as a “language-in-use” capable of expressing and negotiating mathematical ideas (Ávalos, 1996: 82).

The calculator played the role of a mediational tool that gave support to children in making the transition from a step by step strategy to a more relational-based way of working. (p91)

His findings would suggest that disassociation of arithmetic and algebra need not be inevitable.

2.6.4 Children's understanding of letters used in algebra

Many children do not understand letters as numbers, but conceptualise them as objects, for instance, interpreting $6a$ as six apples (e.g. Booth, 1984; Rosnick and Clement, 1980).

Dickson (1989) found that 11-12 year-olds could interpret container symbols, such as \square , as numbers, but did not interpret letters as numbers¹⁶. After all, if $5l$ can mean five times the length of something, why is interpreting $6a$ as six apples not also correct? Confusion about the interpretation of letters is not helped by the fact that letters can be used in many different ways. Graham and Thomas (1999) listed nine different ways¹⁷ in which a letter may be used in mathematics. No wonder students get confused!

Küchemann (1981) listed six different interpretations of a letter, which he linked with Piaget's levels of intellectual development (cf. Lins, 1992). Küchemann's first three interpretations were all ways in which students effectively ignore the algebraic character of the letter: evaluating the letter, ignoring the letter, and interpreting the letter as an object. As students gained in understanding, they interpreted a letter first as a specific unknown, then as a generalised number. His highest level of understanding was that of conceptualising a letter as a variable. Küchemann found that very few students ever reach the highest levels. This framework has frequently been used by subsequent research studies, and now forms a standard for children's conceptual understanding of variables.

¹⁶ In this case, it was not helped by the textbook they were using stating that $10a$ stood for 10 apples.

¹⁷ These are: a name, a placeholder, an index, an unknown, a generalised number, an indeterminate, an independent or dependent variable, a constant and a parameter.

Booth (1984) found that the ‘fruit salad’ approach¹⁸ was not helpful for students (cf. Graham and Thomas, 1999; Tirosh, Even and Robinson, 1998). Once students had grasped that a letter stood for a number rather than an object, Booth found that they tended to conceptualise it as a specific unknown, rather than a variable (1988). This often resulted in students believing that different letters must stand for different numbers, so that $x + y + z = x + p + z$ could never be true (Booth, 1984, 1988; cf. Olivier, 1988). Booth’s results were largely confirmed by the classroom studies reported in this thesis.

2.6.5 Proceptual thinking

If students are to cope successfully with the transition to algebra, they have to learn to think ‘proceptually’. The word ‘procept’ was coined by Thomas (Tall and Thomas, 1991) to express a combination of a process and a concept, and it captures the dual identity of mathematical objects (e.g. Graham and Thomas, 1998). For instance, the symbol $\frac{3}{4}$ expresses both the process of division and a fraction, which is a mathematical object which can be further manipulated¹⁹. Other words that have been used for this include ‘encapsulation’ (Dubinsky and Tall, 1991), and ‘reification’ (Sfard, 1995; Sfard and Linchevski, 1994). Sfard and Linchevski saw mathematical objects as the result of reification, in which processes become permanent entities in their own right. Unfortunately, the same notation is often used for both the process and the object, so it is difficult for the beginner to see that there is a fundamental difference (Gray and Tall, 1994).

¹⁸ a = apple, b = banana, etc., $6a$ means six apples, $7b$ means seven bananas. However, as Graham (1999: 34) pointed out, what does $a \times 2a$ then mean?

¹⁹ While I was teaching, I found that many students were not happy with an answer of $\frac{3}{4}$, but would want to make it into a ‘proper number’ (0.75). They did not see $\frac{3}{4}$ as a number, but as an ‘unfinished’ process.

There is also a tendency for teachers to concentrate on the procedural aspects of algebra as these follow on more naturally from arithmetic, and are easier for students to grasp than the structural aspects (Graham and Thomas, 1998; 2000a).

To cope with the difficult transition from arithmetic process-oriented thinking to versatile algebraic thinking, teaching has tended to emphasise the process side of algebra; the evaluation and manipulation of algebraic expressions. Students have often been taught the rules of algebra so that they could develop the necessary manipulative ability, but with little addressing of concepts. (Graham and Thomas, 2000a: 268)

This is exemplified by the preference for teaching equations initially by 'flowchart' methods²⁰ (a procedural method), rather than by operating on the whole equation (a structural method). Tall and Thomas (1991) referred to this as process management, rather than relational understanding. Tall and Thomas, and Graham and Thomas, used technological tools (computers and graphic calculators respectively) to overcome these limitations.

2.6.6 Misconceptions

As Graham (1998: 6) wrote:

... students significantly under-represent to their teacher the true extent of their ignorance and uncertainty in mathematics.

or to quote Rosnick and Clement (1980: 24):

²⁰ This is a method of solving equations. For instance, $3(x + 2)/2 = 36$ would be solved as follows:

$$\begin{array}{ccccccc}
 & +2 & & \times 3 & & \div 2 & \\
 x & \longrightarrow & x+2 & \longrightarrow & 3(x+2) & \longrightarrow & \frac{3(x+2)}{2} \\
 & -2 & & \div 3 & & \times 2 & \\
 22 & \longleftarrow & 24 & \longleftarrow & 72 & \longleftarrow & 36
 \end{array}$$

At each stage of this process, the operation is written above the arrow. A reverse 'flowchart' then leads to the value of x , by reversing each operation. At no point is the whole equation considered, discouraging holistic approaches, and encouraging purely numerical evaluation.

... large numbers of students may be slipping through their education with good grades and little learning.

Johnson (1989) found that the proportion of students whose progress matched what was intended was very small, and that teachers' views on students' abilities were at odds with the understanding (or lack of it) that their students were able to demonstrate in interviews.

Student ignorance and error can result from a variety of misconceptions. Lee and Wheeler (1989) claimed that the arithmetic/algebra interface was a particularly sensitive point for misconceptions to arise, while Booth (1988) and Nickson (2000) felt that misconceptions were frequently transferred from arithmetic to algebra. Johnson (1989) also reported student errors arising from poor teaching strategies, while Herscovics and Linchevski (1994) speculated whether student failure might reflect the type of instruction rather than the students' learning potential.

In her report on the Secondary Mathematics Project, (1980-83), which was about students' strategies and errors, Booth (1984; 1988) stated that there was a high incidence of errors, and detailed many different error types. One of the most common misconceptions students demonstrate is in their understanding of letters as used in algebra. Indeed, Graham and Thomas (1998; 1999; 2000a; 2000b) believed this to be a significant reason for children's failure to progress in algebra: "one reason that algebra is hard is because the notion of a variable is elusive" (2000a: 266). The concept of a variable underpins algebra, but, they claimed, is rarely discussed in classrooms (cf. Rosnick and Clement, 1980).

Children's errors are not casual or careless, but an indication of "deeply ingrained and resilient misconceptions" (Rosnick and Clement, 1980: 16, cf. Matz, 1980).

Constructivists in particular, believe that "students' misconceptions are never arbitrary or

altogether unreasonable” (Olivier, 1988: 511). Misconceptions are highly persistent and resistant to change through instruction, because they cause students to distort or reject incompatible information, so that they simply cannot ‘hear’ the teacher’s instruction. It is not reasonable to assume that students will overcome such misconceptions by a process of osmosis while learning to manipulate algebraic expressions.

Many children appear to use their own idiosyncratic methods in algebra, often derived from the extension (frequently inappropriate) of methods they have learnt in arithmetic. These methods are often successful initially, but do not enable students to solve harder problems (Booth, 1984; Nickson, 2000). Many difficulties in algebra stem from the use of informal methods in arithmetic which do not generalise or symbolise efficiently (Booth, 1988). The emphasis on correct final answers in arithmetic can often mask the use of an inefficient method, which only becomes apparent much later. Such informal or idiosyncratic methods can be very persistent, being retained after formal teaching has been given (Kieran, 1988a).

I hoped that the graphic calculator model and method of working would enable students to tackle some of their misconceptions. A Piagetian view would anticipate that, if confronted with a conflict between their own ideas and the feedback of the graphic calculator, students would rethink their ideas, and learning would occur. Sometimes, however, when faced with such a conflict, students would ignore it, as in this example (discussed in section 5.2.4) from Claire and Briony’s discussions during the Year 7 case study conducted as part of this research:

Briony: ... equals minus 17, which is a slight problem.

Claire: So I think we got that wrong once again. ... maybe we just wrote down the wrong number.

In this case, the two girls decided that they had written down their predicted answer incorrectly, rather than allowing the discrepancy between their prediction and the result obtained on the graphic calculator to challenge them. Conflict in itself was not always enough: active discussion between students, with the support of the graphic calculator, was necessary for change to occur. This was also found by Ávalos (1996), who observed that students did not always check their answers with the graphic calculator, since they were convinced that the rules they had decided on were correct.

2.6.7 Students' failures at algebra

Accounts of students' failure to learn algebra are widespread. Lee (1996) described students' introduction to algebra as a "cultural shock" (p87), and asked why, if teachers work so hard to make algebra meaningful, do their students find it so meaningless, dislike it so much, and fail so often to succeed (p89). According to Sutherland (1989: 317):

School algebra hopefully provides pupils with a tool, to be used either within mathematics itself or within other disciplines. Unfortunately this potential is not often realised within the school setting.

Few students or adults would disagree with the following remarks made over 75 years ago by Bertrand Russell:

When it comes to algebra and we have to operate with x and y there is a natural desire to know what x and y really are. That, at least, was my feeling; I always thought the teacher knew what they were but wouldn't tell me ... (Bertrand Russell, 1927, cited in Harper, 1987: 86)

Many would also agree with these two students, quoted by House (1988), who were following accelerated courses (implying they were above average attainers):

Algebra is quite hard, and although very educational, it is very frustrating ninety percent of the time. It means hours of instruction that you don't even come close to understanding.

I don't know much about algebra, but who cares?

If this is what good students think, what chance is there for the rest?

Herscovics and Linchevski (1994) felt that students “fail to construct meaning for the new symbolism and are reduced to performing meaningless operations on symbols they do not understand” (p60). Wheeler (1996a) commented that algebra is intrinsically general, abstract and context-free, which is why it is such a powerful tool, but, of course, this is what students initially find so difficult about it.

2.6.8 Cognitive obstacles in learning algebra

In 1978, Davis, Jockusch, *et al* looked at the tasks students were asked to do when they started algebra, and considered that regression because of cognitive overload was very likely. However, they felt that it was not the limitations of students' minds that were the problem, but the form of the learning experience. Herscovics and Linchevski (1994) also queried whether students' failure was a product of the type of instruction rather than their learning potential.

Nevertheless, the possible existence of a cognitive gap or cognitive obstacles between arithmetic and algebra has attracted attention. In 1988, Chalouh and Herscovics viewed students' failure to understand that letters signify numbers, together with lack of closure, the confusion of process and object, and ambiguities in notation, as cognitive obstacles. An ambiguity of notation they mentioned is that 43 means $40 + 3$, $4\frac{1}{2}$ means $4 + \frac{1}{2}$, but $4a$ means $4 \times a$ (cf. Matz, 1980). In 1989, Filloy and Rojano suggested there was a “didactic cut” (p 20) between equations with one occurrence of the unknown and those with occurrences of the unknown on both sides. Herscovics and Linchevski (1994; Linchevski

and Herscovics, 1996) reported that students refused to operate spontaneously on the unknown, and called this a “cognitive gap” (p63).

In 1989, Sutherland agreed that “there is a gap between arithmetical and algebraic thinking which relates to the use of informal methods in arithmetic” (p318). She then found that students would accept lack of closure in a Logo environment, and that they could see that the letters chosen to represent a variable were essentially arbitrary. Two years later (Sutherland, 1991), she said that the idea of a cognitive obstacle or gap needed serious re-examination, and raised the question as to whether such obstacles could be ascribed to classroom practice. Looking at Thomas’ and Tall’s (1988) work using Basic programming, together with work she and others had done using Logo and spreadsheets, she found that students who learnt algebra in these environments did not develop some of the misconceptions about variables as did students taught by more traditional methods. Ávalos’ (1996) work with students using graphic calculators suggests this also.

Sutherland linked the apparent existence of cognitive gaps or obstacles to the Piagetian view. This suggests that language is grafted on to understanding, and so understanding has to be developed first. If students are unable to develop this understanding, it is because they have not yet reached the stage of formal operations, and so are not ready to study algebra. On the other hand, if a Vygotskian view is taken, language becomes a “crucial mediator of inter-psychological functioning and an essential agent in intra-psychological functioning” (Sutherland, 1991: 44). Algebra can then become a language which structures the thinking of the student meeting it, in a process which is essentially dialectical. Sutherland claimed that students’ thinking and problem-solving processes

were moulded by the tools available in the medium they used, such as Logo or a spreadsheet.

It is entirely consistent with a Vygotskian framework, that the nature of the concepts formed, and thus of the misconceptions formed, will be dependent on the tools used in learning. The various cognitive gaps or obstacles identified vary, suggesting that although there may be serious difficulties for students learning algebra for the first time, there is not a specific demarcation between arithmetic and algebra. It seems much more likely that the existence of gaps or obstacles should be ascribed to the teaching method used. The graphic calculator work described in this thesis was originally started with the intention of enabling students to forge links between arithmetic and algebra which would help them to make sense of variables, and to see the connection between their previous learning in arithmetic and the new ideas they were meeting in algebra.

2.6.9 A response to the perceived difficulty in teaching and learning algebra: teaching algebra concurrently with arithmetic

Several researchers have suggested that children could begin algebra earlier than the usual age of about 12 years. Those who believe that learning algebra requires formal operational thinking (as in Piaget's stage theory of development), see no point in teaching it before a child reaches this age (e.g. Küchemann, 1981: 171f). However others believe that algebra is accessible to younger children, although it should not be taught in the same way (Brizuela, Carraher and Schliemann, 2000; Carraher, *et al.*, 2000; Davydov, 1962). Teaching arithmetic and algebra together makes explicit the connections between them, and avoids creating tensions and discontinuities between them:

We are suggesting that arithmetic can and should be infused with algebraic meaning from the very beginning of mathematics education.

... algebraic concepts and notation are part of arithmetic and should be part of arithmetic curricula for young learners. (Carraher, *et al.*, 2000: 2)

Teaching the two together successfully needs the teacher to see how arithmetic and algebra are interwoven, and to draw out the algebraic character of arithmetic. For instance, $3 + 5 = 8$ and $8 - 5 = 3$ will be seen as different types of problem in arithmetic, if the focus is on the operations involved and the answers obtained. When the relationship between the numbers becomes the focus, these can be seen to be the same problem, and the relationship may be considered as algebraic in character (Carraher, *et al.*, 2000). Interpreting the '=' sign as an equivalence relation rather than 'do something' is a similar example of algebraic understanding found in an arithmetic context, for example, in $8 = 3 + 5$.

Davydov (1962) taught an algebraic approach to arithmetic, using letters to stand for numbers, to children aged 6-8 in several schools during 1961-62 in the USSR. He found that they were quite ready to master generalised patterns of quantitative behaviour and to recognise these when written symbolically. The '=' sign was always seen as a symmetric, equivalence relationship in his approach rather than as an instruction to find a numerical answer. The TERC researchers (Brizuela, *et al.*, 2000; Carraher, *et al.*, 2000; Carraher, Schliemann and Brizuela, 2001; Schliemann, Carraher, Brizuela and Pendexter, 1998) conducted a three-year study in which children aged 8-10 were taught algebra concurrently with arithmetic, using letters to signify unknown quantities and to express relationships between them. They found that the children moved gradually from expressing relationships first with natural language, then using iconic representations, drawings and

number lines, and eventually using symbols²¹. These representations became increasingly schematic and context independent (Brizuela, *et al.*, 2000). Similarly, Ávalos (1996) found that the 11-12 year-old Mexican children in his study were perfectly capable of expressing themselves algebraically, using the graphic calculator to provide the language they needed.

In comparing algebra curricula world-wide, Sutherland (2000) found that Japanese children are introduced to algebra before the age of 11, and that they are ready to begin quadratic equations by the age of 11 or 12. She suggested that because Japanese is not an alphabetic language, it is perhaps easier for Japanese children to learn about variables. In Hungary, there is a great emphasis on pre-algebra in primary schools, and the children begin formal work with equations by the age of 10 or 11. It would appear that children's readiness to start algebra is as much cultural as it is developmental (Lins, 1992).

2.7 THE GRAPHIC CALCULATOR AS A MEDIATING INTERFACE BETWEEN STUDENT AND ALGEBRA

2.7.1 Computer mediation

Heid (1996) emphasised the importance of an understanding of a variable to algebraic thinking (cf. Graham, 1998; Graham and Thomas, 2000a; Usiskin, 1988), and used computer technologies to offer a dynamic approach (Heid and Kunkle, 1988). Her focus

²¹ This progression closely matches that described earlier (see section 2.6.2) by those who see the historical development of algebra matched in each individual's development.

was on developing symbol sense (cf. Arcavi, 1994), rather than on symbolic manipulation, with conceptual understanding emerging from the examples used, rather than from definitions. Kieran, Boileau, *et al* (1996) used computer technology so that they could introduce students to variables in the context of functions, rather than simply as unknowns to be evaluated (cf. McConnell, 1988). There was an emphasis on the shift from manipulation as a primary focus to meaning and understanding, as in the work of Thomas and Tall, and Graham and Thomas, discussed below.

Spreadsheets and Logo have found frequent use in this context also. Sutherland described the impact of both Logo (Sutherland, 1989, 1991) and spreadsheets (Sutherland, 1991) on students' understanding of variables (cf. Thwaites and Jared, 1997). She believed that computer environments could allow students to work with richer and more complex mathematical ideas than they usually do when starting algebra. Rojano (1996) used spreadsheets to build on students' informal methods, hoping to avoid the resistance which often occurs when formal methods are introduced. She found that the spreadsheet environment helped students to express their ideas symbolically. Work currently being undertaken at Warwick University by Ainley, Bills and Wilson appears to be showing this also.

Thomas and Tall (cf. Tall and Thomas, 1991; 1988) used programming activities (in Basic) to encourage students to develop their understanding of variables. They modelled a variable as a number stored in a box marked with a letter, and used software which enabled formulae to be evaluated for given numerical values of the letters involved (cf. Thwaites and Jared, 1997). Graham and Thomas (1998; 1999; 2000a; 2000b) used a similar metaphor in their work on the use of graphic calculators for helping students to work with

variables. Ávalos (1996) used the idea of programming a graphic calculator (by putting numbers into a store and evaluating an expression) to encourage children to use algebra as a language which would help them to explore number patterns, and to produce expressions that represent these patterns.

All these researchers found that the use of mental images for a variable was more effective than skill acquisition on its own in enabling students to learn basic algebraic syntax. However, the concerns of the French researchers (noted in section 2.5.3) should be considered here. It is not enough for students simply to work with a computer or graphic calculator. They need also to engage with the technology, in such a way that it does actually mediate between student and algebra.

2.7.2 The graphic calculator model

The 'store' or 'box' model is described by Tall and Thomas (1991): they encouraged students to develop a mental image of a letter as a label for a store. This store could hold a variety of numbers, and could be used in algebraic expressions which could then be evaluated or manipulated on a computer. They felt that this approach showed significant long-term benefits for concept formation, and that concept formation prior to skill acquisition was beneficial for the students in their study. The students attempted to explain and offer reasons for their thinking, and had a more global view of the problems. They also showed a superior understanding of algebraic notation. Thwaites and Jared (1997) also used the 'box' idea, using real boxes with post-it notes on them to name the variable, and strips of card with a value written on them to put into the boxes. They wanted to make a clear distinction between the name or label of a variable and its contents or value.

This idea has been developed by Graham and Thomas (1998; 1999; 2000a; 2000b) and by Ávalos (1996) using the graphic calculator. Unlike computers, graphic calculators are relatively inexpensive, highly portable, and do not need extra software, or programming skills to model a variable. The 26 lettered stores of the calculator form a template for the way that variables act and can be acted upon, while the large screen means that students can see several lines at once, allowing them to reflect on their input and the calculator's output.

Ávalos worked with a group of 11-12 year-olds from Mexico. They were asked to produce 'programs' which would copy tables of input and output values given to them. To do this, they had to work out the function used (ranging from $x \rightarrow ax$ to $x \rightarrow b - ax$), then duplicate it on the graphic calculator. They also had to produce programs for word problems and geometrical patterns. Preliminary conjectures were tested with the calculator, so that the calculator's feedback helped the children to refine their ideas. The children were also able to determine if expressions were equivalent, since they would give the same outputs for given inputs. Through doing these activities they became aware that the letters represented a range of numbers, and that the letters used were arbitrary. They also became used to using algebraic expressions to represent general arithmetic processes.

Graham and Thomas worked initially with 12-14 year-old students from top and middle ability groups from five UK schools, and then with all ability groups from six New Zealand schools. Küchemann's (1981) questions were used, with some others, for pre-and post-tests to determine the students' abilities at standard algebra questions. The module of work lasted about three weeks, which included an introduction to the calculator. It also

included 'screensnaps', which were calculator screens for the students to reproduce. The students used trial-and-error to copy the screens, which then provided:

... consistent feedback [from] which students may predict and test, enabling them to construct an understanding of letters in algebra as stores with labels and changeable contents. (Graham and Thomas, 2000a: 270)

The students doing the calculator modules did not differ from the control students at the pre-test, but by the post-test were doing significantly better in four out of the five UK schools, and in all the NZ schools. Graham and Thomas concluded that the graphic calculator model improved the students' understanding of how letters are used in algebra²². In addition, most of the students and teachers felt this was a much more enjoyable way to study algebra than more traditional approaches.

Graham and Thomas (1999) commented that it should not be expected that a concept like that of a variable would be fully understood on the basis of one short module of work. In particular, they highlighted the fact that a calculator store can contain only one number at a time, whereas a variable can be understood as representing all numbers within its domain at any given time. They also drew attention to the fact that the calculator can only work with rational numbers, and that it cannot therefore use the whole of the real numbers for the domain of the variable. Hence this model has limitations, but:

... if one can manage at this level to assist students in encapsulating the use of letters to represent a discrete subset of the reals which contains, say, rational numbers to N decimal places (where N may be relatively small), then this would be a considerable achievement. (Graham and Thomas, 1999: 20)

²² They concentrated on Küchemann's levels 3 and 4 only – understanding a letter as a specific unknown, and as a generalised number.

Graham and Thomas' work was the starting point for my own classroom work with the graphic calculator. I took their model, and used it in developing my own materials for students to use, particularly the idea of the screensnap (Graham, 1998: 22).

2.7.3 Extending the zone of proximal development with the graphic calculator

Psychological tools are essentially sociocultural in nature, according to Vygotsky (e.g. Kozulin, 1996; Vygotsky, 1978: 57; Wertsch, 1985), being the product of sociocultural evolution. Such tools support a learner in the ZPD, to use Vygotsky's metaphor (the ZPD is the difference between what a child can do unsupported, and what s/he can do with assistance). Jones (1993) suggested that the student in partnership with the technology has the potential to work at a much higher level than s/he would otherwise be able to do. If this is the case, then the technology has the effect of extending the student's ZPD. The research reported in this thesis investigated whether this is so for students using the graphic calculator to facilitate their learning of algebra. The graphic calculator can enable the extension of the ZPD in various ways. It affords a physical model of a variable, and allows students to carry out practical activities using it. It can also furnish language to mediate the learning experience, and this is enhanced if students work together at least some of the time so that discussion can take place.

The interpretation of the ZPD used here is in agreement with that of Meira and Lerman (2001). They criticised views that it is some kind of field or physical space, which the teacher must find in order to teach successfully. Instead they saw it as a "sign-mediated, intersubjective space for analyzing how people become actors and communicators within any given activity or social practice" (p3). Rather than seeing the ZPD as the possession of

the individual, Meira and Lerman interpreted it as a space inhabited by a social group, which is mediated by a sign system such as language. In this space, individuals are enabled to communicate with each other in a meaningful way, so that each can achieve more than they would have done individually. Students working with the graphic calculator, discussing the feedback provided by it as they worked on the various activities given to them, began to establish a common language about variables and to use this to describe something about the nature of variables, as can be seen in the examples in Chapter 5.

The graphic calculator can provide a model of a variable which links the unknown, abstract world of algebra with the known, everyday world of the child, by instantiating the model of a variable as a store for numbers. Students come to school with concepts already in place: “Any learning a child encounters in school always has a previous history.” (Vygotsky, 1978: 84, cf. Cobb, 1991; Nickson, 2000). They need to be able to discuss their existing ideas, and test them out. In particular, students have prior concepts about how letters are used and how they are to be interpreted²³. These prior conceptions get in the way of their learning more viable concepts of a variable. The instant feedback of the graphic calculator helps the learner by validating their ideas (Pratt, 1998) when they are correct, and by giving them privacy when they are wrong. They can experiment in private, trying out anything they like, without having to worry about whether that is going to expose them to adverse attention.

²³ Such as $4a$ meaning 41, or 4, since the value of a is always 1.

The calculator also helps to externalise thinking, which can then be reflected on and discussed, which Pea (1987) saw as a necessary part of what a cognitive reorganiser does. This enables the learner to give meaning to symbols and other mathematical objects, which frequently fails to occur with more traditional teaching methods (e.g. Davis and McKnight, 1980; Pimm, 1987; Sfard, 1991). Unlike manipulatives, where the mathematical content may be less immediately obvious, the graphic calculator displays mathematical expressions directly on the screen. The form in which these expressions are written is the same as those written by hand, thus creating an immediate and obvious link.

The need for discourse in the classroom is well documented (e.g. Dekker and Elshout-Mohr, 1998; Pimm, 1987). However, as Pimm made clear, talk for the sake of it may well not be beneficial: pupil talk needs to be focused (cf. Graham, 1998). Discourse in itself is not the goal, but provides a way to a better understanding of mathematical concepts. To become a successful learner of mathematics, it is necessary to become fluent in the mathematical code (Zevenbergen, 1996). The graphic calculator, in providing mathematical objects on which to operate, also provides objects for discussion, and hence language to aid the discussion. Useful discourse should involve the negotiation of meanings and the sharing of different points of view about mathematics (Pimm, 1987), as the socioculturalists also emphasised (e.g. Laborde, 1990; Lerman, 1996a). Because peer interaction does not involve an authority relationship, negotiation of shared meaning is much more likely to occur when children talk together than in teacher-pupil discussion (Bearison, 1991; Phelps and Damon, 1991).

Oral dialogue enables learners to reflect aloud on new ideas, to verbalise their insights and to resolve conflicts. The connection between dialogue and concept forms an essential

aspect of Vygotsky's theory, as discussed in section 2.4.4. Indeed, Vygotsky went as far as to claim that “[t]hinking in concepts is not possible in the absence of verbal thinking” (Vygotsky, 1987: 131).

2.8 SUMMARY

In this chapter, the literature relevant to this thesis has been reviewed. In sections 2.2 and 2.3, mathematical epistemologies and constructivist theories of learning were considered. The epistemological position taken here is that mathematics is created by human construction, rather than being ‘given’ or in some way already in existence. Knowledge is acquired when it is constructed by an individual, rather than being transmitted to a passive recipient. This accords with the views of both Vygotsky and Piaget, who both held constructivist views. Piaget is perhaps better known in this context, but, as has been seen, Vygotsky also believed that knowledge is actively built up by students, and not passively passed on to them.

The chapter then goes on to consider the life and work of L.S. Vygotsky (in section 2.4). The theoretical position taken in this thesis is grounded in Vygotsky’s work, particularly his theorisation of concept formation through the mediation of tools and signs. His work on the ZPD (zone of proximal development) is also an important way of developing the theory of how students learn with the graphic calculator. The graphic calculator is seen as a cognitive tool (described in section 2.5), which supports a pair of students as they try to understand how letters are used in algebra, and what they mean.

In section 2.6, difficulties in the teaching and learning of algebra were considered, particularly whether cognitive obstacles or gaps exist between arithmetic and algebra. This was followed in section 2.7, by an exploration of how the graphic calculator provides a mediating interface between the student and the algebra.

The purpose of this chapter is to provide a firm foundation and theoretical underpinning for the discussion of the classroom studies presented in the following chapters. The next chapter describes the methodology used for these studies, together with the specific methods of data collection and analysis used.

CHAPTER 3 RESEARCH METHODOLOGY AND METHODS

3.1 INTRODUCTION

This chapter is in three parts. In the first part (section 3.2), general methodological issues, the particular methodological stance underpinning this research and the overall research design are discussed. In the second part (sections 3.3, 3.4 and 3.5), the detail of the research method used is described: this includes the participating schools and students, the methods of data collection, and the analytical methods used. In the third part of the chapter (sections 3.6 and 3.7), the quality of the data and analysis is investigated, and ethical considerations are discussed. Section 3.8 then summarises the chapter.

A brief note at this point is appropriate about the use of language in this thesis, specifically use of the first person, and of active and passive tenses. Quantitative research paradigms are intended to be objective, without the person of the researcher intruding. There is an underlying assumption that the same results and conclusions would have been obtained whoever conducted the research, if it was carried out in the same way. It is therefore traditional in writing up such research for researchers to use the passive tense, and not to use personal pronouns. However, qualitative research is personal: the underlying paradigm assumes that the “knower and the known are inseparable” (Tashakkori and Teddlie, 1998: 10). The methodology I used involved both quantitative and qualitative research, so throughout this thesis I have used the first person and active tenses where it seems appropriate, both to emphasise my own personal participation and to make clear where my

own choices, biases and assumptions are involved. Elsewhere, I have used the passive tense usual in academic work.

The research design for this study is based on a mixed methodology. This chapter starts by giving a brief history of the use of mixed methodologies in educational research. A rationale for the use of such a methodology in this particular research study is then given, together with a philosophical underpinning of the methodology used. This is followed by a detailed description of the overall research design, which comprised two successive stages. The first phase of the design involved a qualitative methodology, while the second phase was a larger-scale survey based on the analytical framework derived from the first phase. The qualitative phase allowed a deep exploration of the research questions in a single school. The follow-up survey gave the opportunity to see if the findings from this school could be generalised to a wider population.

Participating schools and students are then described. Schools were chosen on the basis of accessibility. The initial phase was undertaken in the school in which I was employed for much of that period; other schools participated subsequently if one of their mathematics staff was willing to do so. Selection of participating students is described in section 3.3.

The data collection process is presented in detail for each stage, together with an account of how student samples were chosen from the participants. Qualitative data were collected throughout the first phase. Questionnaires/algebra tests were given to the students before and after the classroom work, to assess their views and skills before and after the teaching modules. Data collection during the classroom work included audiotapes of classroom

discussions between pairs of students, together with their written work, and classroom observation.

The contribution of the data to the evolution of the analytic framework is then explained, and this framework is introduced. Significant themes were derived from the transcripts of the classroom discussions between the students, with the classroom observations contributing a useful form of triangulation for these. The questionnaires/algebra tests were coded to permit analysis adding depth to some of the themes discussed.

The trustworthiness of the data, the analytic process and the reporting are then considered. Quantitative educational research is judged by its validity, reliability and generalisability (e.g. Mills, 2000). Validity addresses the issue of whether the data collected accurately measures what it is claimed to measure. Reliability concerns the consistency of the data over time. Generalisability is about the degree to which findings derived in one setting can be applied to others. These criteria are not easily applied to qualitative research without further consideration, however. Various alternative ways of establishing the *bona fides* of qualitative research have consequently been proposed. Here, a framework derived from the work of Lincoln (1985) is used, and this is discussed. Ethical issues relevant to qualitative research are also considered.

3.2 RESEARCH APPROACH: A MIXED MODEL STUDY

3.2.1 Mixed methodology in educational research

Quantitative, qualitative and mixed research methods are based on different philosophical paradigms. Quantitative methodology has been used in research in many areas for many years. Initially, it was based on a positivist (or logical positivist) philosophy, which holds that “[a]ll genuine inquiry is concerned with the description and explanation of empirical facts” (Mautner, 1997: 438) no matter what the subject matter of the research is. The positivist research paradigm depends on the belief that there is a single reality ‘out there’ which we can discover using our senses. Further, we can agree on what we see, because this reality is independent of the observer (e.g. Bassey, 1999: 42f; Cohen and Manion, 1994: 10; Tashakkori and Teddlie, 1998: 7). *A priori* hypotheses are made, and then tested for their capacity to describe accurately the observed facts.

During the late 1950s, this paradigm became discredited, and was succeeded by postpositivism. Postpositivism was an attempt to address difficulties in the basic axioms of positivism, notably that there is an objective reality which we can discover by observation, but also that research is independent of the researcher and is value-free. Postpositivists believe that, although our understanding of reality is constructed by us, there may still be a reality ‘out there’ to which our constructions approximate. Their experimental method is based on the axiom that causes determine effects, and so it is appropriate to frame hypotheses which can be tested (e.g. Creswell, 2003: 7; Tashakkori and Teddlie, 1998: 7f). Theory is generated deductively, that is, from the general to the particular. Data collected by such researchers tend to be numerical, and are usually analysed by statistical means,

although qualitative data collection and analytical methods can also be based in this paradigm.

During the 1980s, however, postpositivism was rejected by a number of researchers, particularly in the social sciences. Paradigms deemed “more ‘radical’” (Tashakkori and Teddlie, 1998: 9) became increasingly popular, with names such as ‘constructivism’ and ‘interpretivism’. The constructivist (Creswell, 2003: 8; Tashakkori and Teddlie, 1998: 9f) or interpretive (Bassey, 1999: 43; Cohen and Manion, 1994: 36) paradigms are predicated on the belief that there is no independent, objectively knowable reality ‘out there’ at all, and that reality is constructed by us. Our understandings may well be similar, but they cannot be exactly the same, implying that research cannot be independent of the observer. The researcher may also change the situation simply by being there, and/or by asking questions. Because qualitative research depends on observing people in their natural surroundings, it is not possible to make *a priori* hypotheses to test, unlike in the laboratory where a given cause will always produce a certain effect. Instead, many constructivists agree that theory should be generated inductively, from the particular to the general (Tashakkori and Teddlie, 1998: 10). Data collected by constructivist researchers tends to be verbal rather than numerical, and is not normally susceptible to the use of statistical methods, leading rather to qualitative analysis.

More recently still, educational researchers have started using a mix of quantitative and qualitative methodologies. Mixing these methodologies may at first sight seem suspect, since they are based on quite different world-views (Tashakkori and Teddlie, 1998: 11). The paradigms on which they are based have different assumptions about what constitutes knowledge, how research should be done, and how it can be seen to be reliable and valid.

Yet according to Creswell (2003: 4): “[m]ixed methods research has come of age”, with the postpositivist and constructivist paradigms seen as the ends of a continuum, rather than as in opposition and incompatible. Mixed methodology is based on pragmatism:

... the significant issue is not whether one method is overall superior to another but, rather, whether the method a researcher employs can yield convincing answers to the questions that the investigation is intended to settle. (Murray Thomas, 2003: 7)

3.2.2 Rationale for use of mixed methodology in this study

Creswell (2003: 12) argues that researchers should provide a rationale for mixing methods, so what follows in this section provides such a rationale. When I began this research project, I was a mathematics teacher at a girls’ grammar school. The research described in this thesis arose out of work I was doing at the time with my own classes, and which was later taken up by my department. My interest in using the graphic calculator as a model for an algebraic variable, and as a means of teaching algebra, arose when I used it to help a relatively low achieving student to understand better what she was expected to do in her forthcoming SATs exams (as described in 1.2). Immediately after this, I carried out a pilot study with her class. At this stage, my role was as both teacher and researcher. A year later, when I conducted the first part of my main case study, I taught one of the classes involved, the other two being taught by two of my colleagues. At the end of that term, however, I left teaching to work on a mathematics education project in the University of Cambridge. I was able to continue the main case study for a while after this, working with one of my ex-colleagues, and her class. My role by this stage was that of researcher only.

By the end of 2000, I had completed a pilot study, and a two-stage main case study in one girls’ grammar school. The deficiencies of this phase, in terms of the sample of students involved, were obvious, and so I decided to use the teaching method, research instruments

and analytical framework I had developed in the main case study to carry out a larger scale survey in as many mixed, non-selective schools as I could. This constituted Phase II of my research design. Whereas Phase I had used qualitative methods of data collection and analysis, Phase II used the analytic themes developed in Phase I to do a larger, quantitative survey, to see how well my findings from Phase I would generalise to a more representative sample of students.

My methodology was therefore dictated partly by my desire to test out my initial findings on a more representative sample, and partly by the circumstances of my employment (and so was to that extent opportunistic). This combination of a desire for findings which could be generalised and opportunism led me to use a sequential mixed methodology. I believe, however, that this has increased the strength of my conclusions, and contributed important detail.

I am convinced that each research method is suited to answering certain types of questions ... Furthermore, the best answer frequently results from using a combination of qualitative and quantitative methods. (Murray Thomas, 2003: 7)

3.2.3 Methodological underpinning of this research study

Creswell (2003: 5) suggests that three questions should be addressed by researchers.

Firstly, they should clarify the knowledge claims being made, including a theoretical perspective; secondly, researchers should decide on a strategy of enquiry which will be used to inform their procedures; and thirdly, they should decide on methods of data collection and analysis. This section attempts to answer the first two of these questions for the research study described in this thesis. The third question is addressed later in this chapter (3.2.4 and 3.4)

Creswell uses the phrase “knowledge claims” to mean the “theory of knowledge embedded in the theoretical perspective [which] informs the research” (p4), that is, the epistemology which underlies the research. I view the paradigms of the postpositivists and the constructivists as the ends of a continuum rather than incompatible opposites, with the pragmatist taking from each what applies to her/his study. Although instinctively I lean towards the constructivist end of the continuum, I feel that the statistical methods of the postpositivist can add a useful dimension in educational research. Qualitative methods can enable us to look deeply into a situation from which we can determine our analytical themes. Using these in a wider survey permits us to generalise our views to a broader population. As a pragmatist, I wished to claim the benefits of both worlds. This suggested a sequential research design, based on a mixed methodology.

My first strategy of enquiry was that of the case study, and my second was that of the survey. According to Stake (1995: xi) “case study is expected to catch the complexity of a single case”. However, case study¹, as a form of methodology, is not easy to define (Bassey, 1999: 22; Stake, 1995: 2, footnotes 2 and 3). Bassey defines an educational case study as:

... an empirical enquiry which is ... conducted within a localized boundary of space and time ... into *interesting* aspects of an educational activity, or programme ... mainly in its natural context ... in order to inform the judgements and decisions of practitioners ... (Bassey, 1999: 58, original italics)

Stake agrees that case study is specific, bounded, and interesting, and that it occurs in its natural setting (p2). There is an emphasis on studying events in their natural settings in constructivist/interpretive enquiry, rather than setting up carefully controlled experiments.

¹ ‘Case study’ as a general methodological tool is used without an article. ‘A(*the*) case study’ refers to a specific instance of such use.

The constructivist seeks to understand a case in all its complexity, rather than removing complexity, believing that the context is part of the case that is studied.

Bassey sub-divides case study on the basis of function into theory-seeking and/or testing, story-telling, and evaluation. On the other hand, Stake considers the nature of case study: intrinsic case study arises when the researcher wants to learn something about a particular case, instrumental case study when s/he wants to understand a more general phenomenon and uses a particular case to exemplify this. I would like to argue that Stake's instrumental case study is equivalent to Bassey's theory seeking and/or testing category, and Stake's intrinsic case study contains Bassey's story telling and evaluation.

Phase I of this research study contains case study research. This arose initially as an intrinsic part of my work as a classroom teacher: I was trying to find ways of helping my students cope with an aspect of the curriculum they found difficult, and I also wanted to understand better the nature of their difficulties. Initially this led to the work with Sally², and then evolved into the Year 9 pilot study. This stage was evaluative: I needed to justify to myself, my Head of Department, the students and their parents, that using the graphic calculator model would enable my students to make progress. This was certainly found to be the case in the pilot study.

Having established the benefit of the graphic calculator approach, I wanted to look more deeply at its role in introducing children to algebra, and to explore more deeply their

² The girl whose relative lack of ability started my work using graphics calculator to support the teaching of algebra.

understanding of letters, including their misconceptions. This led to the next stage of the study, a two-stage main case study, which was *instrumental* (Stake, 1995) or *theory seeking and testing* (Bassey, 1999) in its form. My intention was to formulate and test theory, and to use these particular case studies to exemplify a more general situation. Initial theory formulation took place during the pilot study. This led into the main case study, which was used to further formulate and test my emerging theories about how children learn algebra, and the graphic calculator's role in this process. My objective was to produce an analysis that would be generalisable beyond the particular situation in which it occurred.

Summarising, Phase I of this research study consisted initially of a pilot study, which was a small, evaluative case study. This was followed by the main case study, which used the themes that had emerged from the pilot study. The main case study was a collective (Stake, 1995: 4) theory-testing case study, focusing on two year groups of students. Phase I produced a detailed picture of these students learning algebra, and the role of the graphic calculator in their learning.

The purpose of Phase II was to use a survey to see how far my conclusions could be generalised to a wider population. According to Creswell (2003: 14), surveys can include "cross-sectional ... studies using questionnaires ... for data collection, with the intent of generalising from a sample to a population". Since Phase I of this study was carried out in a girls' selective school, generalisation to a wider population was questionable without further data collection. I hoped that a survey of students in more representative schools would allow such generalisation.

3.2.4 Overall research design

The overall design is a mixed model study, which is sequential (Creswell, 2003: 16; Tashakkori and Teddlie, 1998: 46f). According to Tashakkori and Teddlie, mixed model studies are the product “of the pragmatist paradigm and ... combine the qualitative and quantitative approaches within different phases of the research process” (p19).

My study is two-stage: the first phase was mainly qualitative, followed by a second phase consisting of a follow-up survey, with its analysis dependent on the themes and categories identified in the first phase. This research design allowed an initial exploration using qualitative data to produce a thematic analysis, rich in detail, which was then extended through a larger, more representative survey. Table 1 (on the following page) gives details of the two phases, the case studies comprising Phase I, and the data collection involved.

3.3 PARTICIPATING SCHOOLS AND STUDENTS

3.3.1 Phase I participants

The pilot study and main case study, which comprised the first phase of this research study, were all conducted at one selective girls' grammar school just over 40 miles to the north-west of London. At the start of this study in 1999, the school had around 1000 pupils on roll. Entrants to the school were selected by their performance in standardised

Table 1: Research Design

	<i>Phase I</i>			<i>Phase II</i>
	<i>Pilot Case Study</i>	<i>Main Case Study</i>		<i>Survey</i>
		<i>Initial stage</i>	<i>Follow-up stage</i>	
<i>Date</i>	June/July 1999	Oct/Nov 2000	Oct/Nov 2001	Spring/Summer 2002
<i>Age of students</i>	13-14 years, Year 9	11-12 years, Year 7	12-13 years, Year 8	10-13 years, Years 6-8
<i>Sex of students</i>	Girls	Girls	Girls	Mixed
<i>Types of schools</i>	Selective	Selective	Selective	Non-selective
<i>Number of schools</i>	1	1	1	4
<i>Number of classes</i>	1	3	1	12
<i>Number of teachers</i>	1 ³	3 ⁴	1	6
<i>Number of students</i>	30	79	28	307
<i>Data Collection</i>	<ul style="list-style-type: none"> ◆ Audiotapes of classroom discussion (5/6 pairs of students⁵) ◆ Written work of sample students ◆ Classroom observations ◆ 6 × student interviews (pre and post) ◆ Questionnaires (pre and post) ◆ Algebra tests (pre and post) ◆ 10 student journals ◆ My journal 	<ul style="list-style-type: none"> ◆ Audiotapes of classroom discussion (3 × 3 pairs of students) ◆ Written work of sample students ◆ Classroom observations ◆ Preliminary student interviews as required ◆ 3 × 3 student interviews (post) ◆ 2 × teacher interviews (the following July) ◆ Questionnaires (pre and post) ◆ My journal 	<ul style="list-style-type: none"> ◆ Videotapes of whole class discussions (pre and post) ◆ Audiotapes of classroom discussion (3 pairs of students) ◆ Written work of sample students ◆ 2 × student interviews (post) ◆ Teacher interview (post) ◆ Questionnaires (pre and post) ◆ My journal 	<ul style="list-style-type: none"> ◆ Questionnaires/ (pre, immediate post and delayed post) ◆ 4 × teacher questionnaires
<i>Type of analysis</i>	Mainly qualitative, plus some descriptive statistics	Mainly qualitative, plus some descriptive statistics	Mainly qualitative, plus some descriptive statistics	Mainly descriptive statistics based on qualitative analysis in previous stage.

³ This was my class, and I was acting as both teacher and researcher at this stage.

⁴ I was one of these teachers, again acting as both teacher and researcher.

⁵ Two students missed the middle two lessons.

tests (although these did not include a mathematical component) and, up to August 2000, entered the school at the age of 12 years (Year 8). The number of students on roll rose to about 1250 once Year 7 students were admitted from September 2000. The school prided itself on its excellent academic record with over 95% of students achieving five or more A* to C grades at GCSE⁶ in any given year, and 99% achieving level 5⁷ in the mathematics SATs⁸. These statistics are all well above those for the Local Education Authority of which the school is a part, and whose statistics are themselves well above the average for England and Wales as a whole. Less than 5% of students are on the register for special needs, with no more than one or two students in the school with a statement of special needs at any one time. Academically, this was a privileged school whose students could expect to gain excellent results.

The class that took part in the pilot study was in Year 9 at the time, and comprised 30 students aged 13-14 years. All these students gained A* to C grades in mathematics GCSE two years later, except for Sally, who achieved a grade D. I had taught this class mathematics since they entered the school in September 1997. The participants in the first stage of the main case study, which took place in the autumn term of 2000, were three classes (79 students) of Year 7 students, aged 11-12 years. These students had entered the school just a month or so before the case study began. One of the classes was taught by me, the others by two of my colleagues. The follow-up to this initial stage of the main case study was done with just one of these classes in October/November 2001, when they were

⁶ General Certificate of Secondary Education, the examinations taken by 16 year old students at the end of Year 11.

⁷ The target grade.

⁸ Standard Assessment Tests for students at the end of Key Stage 3 (the end of Year 9 when students are aged 14 years).

in Year 8 and aged 12-13 years. This was the only class which kept the same teacher as in the previous year. In order to reduce the number of different factors affecting the students, I felt it was important to focus on a class where such continuity occurred.

Student samples

At each stage of Phase I, a sample of students audiotaped their discussions during classroom work while they used the graphic calculator, and allowed their written work to be used to clarify the verbal record. In the pilot study, there were initially six pairs of students in the sample. However, a trip to France intervened, and one student from each of two of the pairs missed the two middle lessons of four because of this. The two students left formed a pair, so that five pairs of students audiotaped their work during these two lessons. Even with this absence, a third of the class were involved in recording their conversations as they worked, giving a good spread of data from the classroom sessions.

All these students were volunteers and there were no other criteria for choosing this sample, other than asking volunteer students to work with someone of similar mathematical aptitude to themselves. This stipulation was made because I wanted to see if the graphic calculator would support the students in making cognitive gains, and I felt that if one student was significantly ahead of the other, genuine collaboration would be unlikely to flourish between the two students as they worked together with the calculator. The students were asked to talk about what they were thinking, and to state what they were doing with the graphic calculator, as they pressed the keys. This means of collecting data proved very successful, giving me some access to the students' thought processes. It was clear that any shyness or undue awareness of the tape recorder was soon overcome, and using it became routine for the students. However, it should be recognised that articulating

their thoughts to each other for the tape would have benefited the students' learning, and that effects for these particular students cannot therefore be solely ascribed to the graphic calculator.

Because this had proved a useful means of gaining some access to the students' thought processes as they worked with the graphic calculator, it became a major aspect of the data collection for the main case study. However, I wanted to choose the student sample in a less arbitrary way. Anticipating (correctly) that most of the Year 7 students would want to be part of the samples, I asked the students in both my class and that of one of my colleagues to work in pairs on a short problem, recording their discussions as they tried to solve it. I used these recordings to choose three pairs from each class on the basis of the students' ability to describe clearly what they were thinking. I also tried to ensure that the three pairs from each class covered a range of mathematical achievement, basing this on teacher assessment. The three pairs from the third class were chosen by their teacher using the two criteria that students should be able to talk fluently about their work, and that the three pairs chosen should demonstrate a spread of mathematical achievement. Preliminary recording tests were not carried out with these students, because they joined the study at a later point than the other two classes, and time was short. Three pairs from each class were chosen, so that nearly a quarter of the students from each class (containing around 26 students) were recording their work, again giving a good spread of data.

In the follow-up stage of the main case study, the three pairs of students who had recorded their work the previous year again recorded their work. This ensured that there was continuity, and that conversations from one year could be directly compared with those from the other.

3.3.2 Phase II participants

In the second phase of my research study, which consisted of a survey, four different schools participated. These were chosen on the basis of accessibility: I contacted all the teachers I was by then in contact with through my employment, asking if they would be willing to take part in this survey. Seven teachers initially responded, and four of these actually participated, one also involving two of her colleagues.

The schools

Details of the four schools involved are summarised in Table 2⁹ (on the following page). It can be seen from Table 2 that a spread of students from different areas of the country and from different year groups was included in this survey. The middle school, school *B*, used the graphic calculator approach and materials with about half their students, whereas the other three schools used one or two classes only. Originally seven teachers were involved, one at each of schools *A*, *C* and *D*, and four at school *B*. One of the classes at school *B* did not do the delayed questionnaire however, and so I decided to omit data from this particular class completely, which meant that six teachers and 12 classes were included in the final analyses.

⁹ All information relates to the year 2002.

Table 2: The schools participating in the Phase II survey

	<i>School A</i> ¹⁰	<i>School B</i>	<i>School C</i>	<i>School D</i>
<i>Area of England in which school is located</i>	South coast	East Anglia	South coast	North east
<i>Nature of the school catchment area</i>	Mainly rural, on outskirts of medium-sized town	Very rural area	Medium-sized town near large city	Small satellite town on outskirts of major industrial city
<i>Type of school</i>	Mixed, comprehensive	Mixed, middle school	Mixed, comprehensive	Mixed, comprehensive
<i>Age of students at school</i>	11 to 16	9 to 13	11 to 16	11 to 18
<i>Number of students on roll</i>	1002	427	1014	1084
<i>% of students on the SEN</i> ¹¹ <i>register</i>	22.5%	13.8%	18.9%	4.6%
<i>% of students with statements of special needs</i>	1.0%	6.1%	0.6%	1.2%
<i>% students achieving target grade in end of Key Stage mathematics tests</i> ¹²	69%	68%	86%	63%
<i>Number of students participating in this study</i>	41	214	22	30
<i>Number of teachers involved in this study</i>	1	3	1	1
<i>Number of classes participating</i>	2	8	1	1
<i>Year groups included</i>	Y8	Y6, Y7, Y8	Y8	Y7

All four schools were e-mailed general instructions for teaching the module and collecting data. In addition, detailed teachers' notes on the graphic calculator model and how the calculator was to be used in the classroom, and a set of worksheets to use over a period of about three hours of lessons, were also sent to the schools¹³. The actual time taken for the teaching and the specific materials used were decided entirely by the teachers. All data

¹⁰ No order of any kind is intended by the use of the letters *A*, *B*, *C* and *D* to denote the schools.

¹¹ Special Educational Needs

¹² Level 5 is the target grade for the end of Key Stage 3 Standard Assessment Tests, taken when students are aged 14 (and applies to schools *A*, *C*, and *D*). Level 4 is the corresponding target grade for the end of Key Stage 2 SATs, taken when students are aged 11 (and applies to school *B*).

¹³ All these resources can be seen in Annex I.

collection was done by the teachers concerned, and I was not involved beyond providing materials and instructions.

The students

A total of 307 students were involved in this survey, distributed as shown in Table 3:

Table 3: Distribution of age groups of students across schools

<i>Year group</i>	<i>School A</i>	<i>School B</i>	<i>School C</i>	<i>School D</i>	<i>Sub-totals</i>
<i>Y6</i>	-	(36) 11.7%	-	-	(36) 11.7%
<i>Y7</i>	-	(109) 35.5%	-	(30) 9.8%	(139) 45.3%
<i>Y8</i>	(41) 13.4%	(69) 22.5%	(22) 7.2%	-	(132) 43.0%
<i>Sub-totals</i>	(41) 13.4%	(214) 69.7%	(22) 7.2%	(30) 9.8%	(307) 100%

The students (Year 8) from school A were in two different sets, both taught by the same teacher. The higher achieving group, which was the second set of five (although their teacher¹⁴ qualified this by stating that students were only roughly set in Year 8), was described by the teacher as of:

Wide ability. KS2 SAT average¹⁵ = 4. Probably working at good 5. Will all follow intermediate [GCSE level]. Interested and enthusiastic, but not 'natural' mathematicians.

Their previous algebra experience was "mostly 'pre-algebra'", and their understanding of letters was varied, with most "probably hav[ing] a 'formal'/learned, rather than organic understanding". The same teacher described the lower achieving group at school A (the fifth set of five) as:

Poor and lacking confidence on the whole. KS2 level 3 mean¹⁶. Probably working at level 4 mostly¹⁷.

Their previous experience of algebra was also described as “pre-algebra”, and their understanding of letters as “[p]oor.”.

The school *C* students (Year 8) were in the seventh set of seven, and were described by their teacher as “[v]ery weak relative to their year. Main problem is concentration and retention.” Asked what algebra they had studied previously, the teacher’s response was:

Obviously some in year 7, but they are taught mixed ability in Y7, so possibly little stuck at all. A bit in Y8 using algebra to describe simple sequences.

In Year 8, the teacher characterised their understanding as still “[p]oor. Typical misunderstandings shown – e.g. $5x$ when $x = 3$ is 53.”¹⁸

The school *D* students (Year 7) were in the top band of two bands in their year group, and were described as “Generally good. End of KS2 SATs results¹⁹, 4-5.” They had “[v]ery little [previous experience of algebra] except for finding rules from patterns”, and their understanding was characterised by “[s]ome confusion based on a mixture of previous knowledge”.

The students (Years 6, 7 and 8) at school *B* were divided into upper and lower bands in each year group. The year 6 students were from the upper band in their year group, plus

¹⁴ Information about the classes came from the questionnaires completed by the contact teacher at each school.

¹⁵ Key Stage 2 Standard Assessment Tests, taken at age 11. Level 4 is the target grade for these Tests.

¹⁶ Their mean score on the KS2 tests was Level 3, so below the target grade for their age.

¹⁷ So assessed by their teacher as about 2 years behind the target grade for their age.

¹⁸ This is the ‘code’ error described in more detail in Chapter 7, *Misconceptions*.

¹⁹ So at or above the target grade for their year group.

five relatively good students from the lower band. Their teacher described them as “a few L6 candidates, rest mixture L4 and L5”. This described the levels at which these students would be entered for the KS2 SATs²⁰ at the end of that academic year. The Year 7 students were in a variety of groups. Some were mainly working at Level 4 at the time of this study while others were working at Levels 5 and 6. Some of the Year 8 students were revising Level 4 and 5 topics, and others were working at Levels 6 and 7. Their experience of algebra was described as “little” for the Year 6 students, “some” for the Year 7s and “quite a bit” for the Year 8s. Their understanding of letters prior to this module was described as “quite good” for the Year 6 and 8 students, but “patchy, especially amongst the less able” for the Year 7s.

Information about the level of achievement of the students at all the schools, as given by their teachers, is summarised in Table 4:

Table 4: Distribution of achievement levels²¹ of students across schools and year groups (figures give numbers of students)

Year group	School	Level 3	Level 4	Level 5	Level 6	Level 7
Y6	B		← 36 →			
Y7	B		53	← 56 →		
	D			30		
Y8	A		10	31		
	B		← 44 →		← 25 →	
	C	← 22 →				

²⁰ Target grade Level 4.

²¹ These levels are National Curriculum levels as assessed by the teachers of the classes.

Table 4 shows that the students participating in the survey covered a good spread of achievement levels and age groups.

3.4 METHODS OF DATA COLLECTION

3.4.1 Data collection in Phase I

Throughout the first phase, data of a mainly qualitative nature were collected. These included audiotapes of students' discussions while working with the graphic calculator, students' written work, teacher and student interviews, and questionnaires testing students' ability to answer standard algebra questions and asking about their understanding of letters. Other forms of data were collected at certain stages of this phase, including videotapes of classroom discussions and student journals. In addition, I kept a research journal in which I noted my observations and reflections. The most valuable data sources were the transcripts of the students' discussions during lessons and the questionnaires. However, collecting so many different types of data meant that the information contained in one data source could be triangulated against that found from other sources.

Data collected during lessons

Transcripts of students' discussions

As described earlier, a subset of students from every class involved in the three case studies recorded their discussions as they worked with the graphic calculators. These were transcribed, providing a rich source of primary data concerning the development of the students' thinking while they worked in their pairs with the graphic calculator.

Students' written work

The students audiotaping their discussions did all their written work in special notebooks during the pilot study and the Year 7 stage of the main case study, which I kept as part of the data collection. The Year 8 sample did their work in their normal exercise books, which I then photocopied so I had a record of their work also. In all three cases, this proved valuable in supplementing the oral record from the audiotapes, both in ensuring I interpreted the oral record correctly, and in giving further evidence of how the students worked on questions.

Classroom observations

One of my colleagues observed one of my lessons during the Year 9 pilot study, and discussed her observations and reflections with me afterwards. During the Year 7 study, I observed lessons in my colleagues' classes, and one of them observed during my lessons. These observations included discussions with individual pairs of students while they were working, as well as observation of whole class discussion. Consequently they helped to establish how the students were using the calculators, and gave opportunities for probing students' understanding of letters and the misconceptions they held, thus triangulating with the data obtained from questionnaires, classroom transcripts and student and teacher interviews.

Video Data in the Year 8 Case Study

In lieu of classroom observations in the Year 8 stage of Phase I, as I was no longer working at the school in question, I asked the class teacher to conduct two 20 minute discussions with her class. These took place before and after the classroom work, and were based on questions I had picked out from the two questionnaires the students did. I videotaped the discussions, and then transcribed them. Like the classroom observations, these were useful for triangulating with other forms of data. Being personally present

during these discussions also helped me to contextualise the other data forms, which were collected by the class teacher.

Questionnaires/algebra tests²²

The questionnaires/algebra tests gave a snapshot of the students' views and their ability to answer standard algebra questions before starting the classroom work and after it finished. They also provided evidence for the misconceptions students held.

In the Year 9 pilot study, separate questionnaires and algebra tests were given to the students to complete both before and after their work with the graphic calculator. The questionnaires concerned their attitudes to mathematics, to algebra and to the graphic calculator, which they had used previously for other topics. The post-questionnaire also concerned their attitudes to the work they had just done. The algebra tests contained questions designed to probe the students' ability to deal with algebraic expressions and equations, with the first algebra test taking about 30 minutes and the second one an hour. When I analysed these, I felt that a lot of the information I had collected did not help me to gain an understanding of the students' interpretation of letters, and was therefore redundant for research purposes (although useful for teaching purposes). I also felt there were too many questions, with the effect that the students undoubtedly felt that these were assessment tests, rather than a means of my gaining information about their understanding.

Consequently, when I planned the initial stage of the main case study for the Year 7 students, I decided to combine the questionnaire and the test, so that the students had a

²² All questionnaires and algebra tests can be seen in Annex III.

single questionnaire to complete. This contained a few questions testing their understanding of letters, and their ability to answer standard questions, plus other questions asking what they understood letters to mean, and about their previous experience of algebra. When I came to analyse these, I felt they gave adequate information to enable me to make worthwhile inferences about the students' understanding of algebra and letters, and their ability to do simple questions. These questionnaires took the students about 20 minutes to complete, and so were perceived in a much less threatening way. In the follow-up stage of the main case study with the Year 8 students, I used more algebra questions, and omitted the questions about previous experience, since this was already known to me. Some of these algebra questions repeated those in the Year 7 questionnaire using different letters and numbers, but the same structures. Again, these took about 20 minutes to complete.

When it came to planning Phase II, I decided to work with questionnaires similar to those I had used in the main case study. These had proved to contain a lot of information, which could be analysed in a number of different ways. In particular, they had enabled me to gain some understanding of the students' interpretation of letters, their ability to answer standard questions, and their misconceptions. The questionnaires used in the survey were therefore similar to those of the main case study, with a little information requested about students' attitudes to algebra and using the graphic calculators.

Interviews²³

All the interviews were semi-structured. In each case, I decided on a set of questions I wanted to ask, but used these to develop a conversation with the students and teachers to explore their thinking on the issues raised. I was aware that in the interviews with the students, I was in a position of authority and that this could colour how they responded. However, I tried to mitigate such effects by emphasising that the interviews would only be heard by me, and that nothing they said would be used either by me, or by any other teacher, in any way other than as a source of data for this research study. I also tried to ensure that the questions were not ones which would cause them any embarrassment or difficulty, other than the difficulty of answering questions about algebraic expressions or equations.

Student interviews

All the student interviews lasted about 15-20 minutes. In the pilot study, all the students who audiotaped their work were interviewed in their pairs after the first questionnaire and test, but before the classroom work, and then again after the second questionnaire and test, at the end of the whole study. In their first interviews, students were asked about their attitudes generally to algebra and mathematics, then about how they had answered specific questions from the test. Discussing these answers then led into an exploration of their understanding of letters. In the follow-up interviews, students were asked about their attitudes to using the graphic calculators, and questions from the follow-up test were used to stimulate further exploration of their understanding of letters. I also asked them what difference they thought recording their conversations had made.

²³ Interview schedules can be seen in Annex II.

When it came to the Year 7 stage of the main case study, I decided to do a few brief preliminary interviews with individual students purely to probe their answers to the first questionnaire. Students were chosen for these interviews if they had written a response to the question about their understanding of letters which I did not understand, or which I wanted to explore further. Follow-up interviews were held with all the pairs of students who had audiotaped their discussions during lessons. These interviews were used to investigate the students' understanding of letters, specific misconceptions which arose or became apparent during the teaching period, and their views on using the graphic calculator.

In the Year 8 follow-up stage of the main case study, I did not do any preliminary interviews with the students, partly because I was no longer at the school on a daily basis, but also because I had decided to video a whole class discussion which enabled me to get some idea of the whole class's understanding of letters at this point. After the classroom work was finished, I returned to the school to do a second videotape of another whole class discussion, and to do follow-up interviews with two groups of students. Although I had asked the same students to record their work as in the previous year for the sake of continuity, I wanted to get some idea of what the rest of the class thought as well. I therefore asked the class teacher to choose two groups of four reasonably articulate students, one group chosen because they were high achieving relative to the class as a whole, and the other chosen because they were relatively low achieving. Each group was asked the same questions, which covered a range of algebraic questions including their understanding of letters, how they used the graphic calculator and their attitudes to using it.

For the pilot study and Year 7 interviews, when I talked to the sample students in their pairs, I recorded each interview and made notes, particularly on mathematical aspects of the conversation. For the Year 8 interviews, where I was talking to groups of four students, I put questions onto pieces of card, put them face down on the table, and then asked students to pick a card. Each student took it in turn to do this, answering the question on the card. I then opened up each question to discussion with the whole group. I found this a good way of ensuring that everyone had a chance to speak, and that no one person dominated. Again, I audiotaped the interviews, and took notes of mathematical aspects.

Teacher interviews

In the pilot study, I was the only teacher involved, and I recorded my observations and reflections in my research journal, together with those of the colleague who observed one of the lessons. In the Year 7 stage of the main case study, two of my colleagues were also involved. Initially, I intended to interview them at the same point that I interviewed the students after the class work had finished. This proved difficult to arrange, and I felt there would be benefits in interviewing them later in the year. As it happened, I left the school not long after finishing this stage of the main case study, and was not able to interview them until the end of that school year in the following July. This proved useful, however, in that as well being able to reflect on the specific module in the case study, they were also able to consider how the students' use of the graphic calculators to start their study of algebra had impacted on other algebraic topics they studied during that year.

I held separate interviews with the two teachers, lasting about 45 minutes each. I audiotaped these and took supplementary notes, particularly of mathematical detail. The questions used to initiate discussion covered four areas: the classroom work, the students' understanding of letters and algebraic syntax, types of misconception shown by the

students and the teachers' own views on the feasibility of the graphic calculator model as a teaching method. The discussion of the classroom work included how the teachers introduced the graphic calculator to their students, whether they used their own material in addition to the worksheets I had prepared and how they would improve those worksheets. We also discussed algebraic topics they had covered subsequently with their classes. The next set of questions was about how useful the graphic calculator was in helping the students to interpret letters appropriately and to learn algebraic syntax, and if they thought the graphic calculator model could be used with younger students. I then went on to ask them if they had observed specific misconceptions I had found in the data, and whether they were aware of other misconceptions. I also asked if they had realised that the students were sometimes using the graphic calculators in ways which would merely reinforce errors, rather than correcting them. The final area of discussion concerned their willingness to take part in the next phase of the main case study, if they had a suitable class the following term.

After the Year 8 phase of the main case study, I interviewed the teacher involved at the same point that I interviewed the students, which was soon after the classroom work finished. Again, I audiotaped the interview and took supplementary notes of mathematical detail. There were three sets of questions in this interview, which lasted about 40 minutes. The first set consisted of general questions about the teaching module, and how the students used the graphic calculator during it. The second set concerned specific algebraic questions I had asked the students about in their interviews, and how the teacher thought they might approach them. The final set of questions was about methods students might use for certain types of question, and how the graphic calculator could have helped them with these.

Journals

Student journals

In the pilot study, I asked my student volunteers to make journal entries, describing their reactions to the lessons and how they felt they coped with the work and the graphic calculator. These students had special notebooks to do all their classwork and homework in together with these entries. I asked the students to make their comments during the final five minutes of their homework, so they were not doing it in time which they would have regarded as their own. However, overall these entries proved to be disappointing. There were a few useful comments, but most were very brief, not providing any extra insights. I therefore decided not to use this form of data collection in future studies.

My journal

I kept a research journal throughout my research study. During the pilot study and the main case study, I was able to use this to make observations and to reflect, both on the conduct of the studies and on incidents I saw and comments I heard in the classroom. During the Year 8 case study, I used it to record thoughts that I wanted to check with the class teacher or the students, and to record conversations with the class teacher. This has proved useful in providing my immediate reactions to events, and in checking my memory.

3.4.2 Data collection in Phase II

Student questionnaires²⁴

In Phase II, the main method of data collection was three questionnaires given to all the students to complete. The first was done before the classroom work, the second immediately afterwards, and the third some four to six weeks later. All three contained the same questions, except for varying the precise numbers and letters used, so that they could all be directly compared with each other. The questions were all ones which had been trialed during one or more stage of the main case study, so that further comparisons could also be made. Students were also asked whether they had found the classroom work difficult (on a scale of 1 to 5), and how helpful using the graphic calculators had been (again on a scale of 1 to 5).

Teachers' questionnaires

The main contact teacher in each school involved was asked to complete a short questionnaire, consisting of open questions. This was intended to establish teachers' assessment of their class(es)' level of achievement in mathematics and algebra, whether they had used any material of their own in addition to the worksheets provided, and their opinion of both the graphic calculator model and the teaching materials.

²⁴ All questionnaires can be seen in Annex III.

3.5 DATA ANALYSIS

Phase I provided the main analytical tools used in both phases of this research study. Four major themes emerged from the classroom transcripts and the questionnaires, and, to a lesser extent, from the other data. These were the students' understanding of letters, their ability to answer simple algebraic questions, any evidence found of cognitive change during the classroom work, and the students' misconceptions. The questionnaires from the Year 7 stage of the main case study were used to provide an analytical structure which was then used for the Year 8 questionnaires and, retrospectively, for the pilot study questionnaires. Analysing the questionnaires was an iterative process with many iterations. Once I was satisfied with the structure I had devised, I used it to analyse the Phase II survey questionnaires.

3.5.1 Analysis in Phase I

The pilot study

My initial analysis of the pilot study materials was aimed at establishing whether the teaching module provided a satisfactory way of helping students grasp various aspects of algebra, specifically simplifying algebraic expressions and solving linear equations. For this, I simply compared marks gained before and after the teaching module. As well as evaluating the teaching method, I also wanted to see if there was any evidence that the graphic calculator acted either as an amplifier, speeding up the learning process but not producing any fundamental changes in it, or as an agent of cognitive change (Berger, 1998;

Pea, 1985, 1987). In addition, I looked at the students' attitudes to using the graphic calculator.

Two major themes which emerged from analysis of the classroom data in the pilot study were students' developing ability to answer algebraic questions appropriate to the stage they had reached and cognitive change in their thought processes. Evidence for the students' ability to answer algebraic questions before and after the teaching module came from the algebra tests they did. The evidence of cognitive change came from the classroom transcripts, supported by the students' written work and their interviews. As a teacher, I found myself surprised by the sophistication and depth of some of the students' discussions as they worked. As a researcher, I found myself surprised to see that the graphic calculator rarely acted as an amplifier: indeed, it often slowed the students down, rather than speeding them up. I was even more surprised to see strong evidence for cognitive change. This was quite the opposite to Berger (1998), whose work had provided a stimulus for this study (section 2.5.4).

Reflecting on this, I felt that there were in fact two crucial questions about which I wanted to be able to say something. One was whether the graphic calculator could be an agent for cognitive change. The other issue was the students' interpretation of letters. Examining the classroom transcripts and other supporting data allowed me to comment on the role of the graphic calculator, but I found little I could say about how the students understood letters at this stage. As I moved on to the main case study, I therefore decided to focus the questionnaires on this issue of student understanding of letters, and to see again if evidence of cognitive change in this understanding would emerge from the classroom data. I also

decided to focus my next case study on Year 7 students, who had not yet started any formal study of algebra, to see what interpretations they held prior to doing algebra.

The main case study

As a result of my findings in the pilot study, when I started the Year 7 study I planned a questionnaire which I hoped would reveal something of the students' understanding of letters. I also hoped that collecting other forms of data, particularly audiotapes of the students' discussions while working, would again provide evidence of cognitive change.

The classroom data of all kinds provided a very rich source of analytic themes. These included the learning environment provided by the graphic calculator, the students' interpretation of letters and their proceptual understanding²⁵, evidence of cognitive change, the role of discussion, and the students' misconceptions. Considering these enabled me to conceptualise the role of the graphic calculator, by using Vygotsky's (1978: 86) zone of proximal development (ZPD, section 2.4.4). It appeared to me that a pair of students working together with a graphic calculator created a ZPD (sections 2.7.3 and 4.3). The combined elements of discussion and the graphic calculator's instant, neutral feedback provided the support which enabled each student to reach a higher level than either would have done on their own and without a calculator. The evidence for their achieving higher levels of understanding came from the analyses reported in Chapter 6: *Developments in students' understanding and skills*.

²⁵ A student who thinks proceptually (Tall and Thomas, 1991) can understand and use, e.g. $x + 1$, as both a mathematical object in its own right, and as the result of a process. A student who does not think proceptually only sees the process.

The theme of misconceptions (Chapter 7: *Misconceptions*) emerged primarily from the questionnaires, but also from the classroom data, particularly the observations. I used the algebra questions on the questionnaires to classify students' incorrect answers. Several major categories appeared, which included substituting specific numbers for letters (e.g. $a = 1$, $b = 2$, $c = 3$, and so on); interpreting letters as individual digits of a number (e.g. $6c = 63$); interpreting letters as arithmetic processes (e.g. a means $+1$, b means $+2$, and so on); interpreting letters as exponents in some way (e.g. $2a$ means 2 squared, $2b$ means 2 cubed, and so on); ignoring the letter completely (e.g. $6c$ used as just 6); and interpreting letters as objects (e.g. $6a$ means 6 apples). These interpretations were further explored using an open question asking students directly what they understood the letters to mean. In cases where I was still unclear, I used the interviews to investigate students' views further. Other misconceptions became apparent during the classroom observations, and I discussed these with both students and teachers during the follow-up interviews.

I used the analytical structure which I derived from the Year 7 stage of the main case study for the Year 8 follow-up study. This required reconsidering all the themes and categories, and revisiting the Year 7 data. Finally, I re-analysed all the data a third time, this time including the Year 9 pilot study data. In this way, the analytical structure I devised based on the Year 7 data was reviewed on three separate occasions, as well as repeatedly during each revision, each time checking that the themes I identified were representative of the data.

3.5.2 Analysis in Phase II

The analytical themes and categories I had created for the questionnaires in Phase I were then used for the analysis of the survey. This required a further review of the categories

used to code the questionnaires, and all questionnaire data, both from the case studies and the survey, were revisited yet again. In the survey, two open questions were asked about students' interpretation of letters, and how they had actually worked out one of the algebra questions. The combination of these two proved very useful in giving evidence of students' thinking about letters, and how they worked with them.

3.6 QUALITY OF THE DATA AND INFERENCES

Reliability is the extent to which a research fact or finding can be repeated, given the same circumstances, and validity is the extent to which a research fact or finding is what it is claimed to be. (Bassey, 1999: 75).

It is usually not possible in qualitative studies to demonstrate reliability and validity since case study and other qualitative research depend on a specific context. Repetition of a study is impossible, since the exact circumstances can never be repeated. As an alternative to reliability and validity therefore, the concept of 'trustworthiness' was established by Lincoln and Guba (1985) as an appropriate substitute or analogue for qualitative methodologies (Bassey, 1999: 75; Tashakkori and Teddlie, 1998: 90). This concept will be used as a way of checking the standard of Phase I of this research project. As Phase II uses the analytical framework developed in Phase I it will also be assessed in the same way.

3.6.1 Trustworthiness

Lincoln and Guba introduced four criteria which give qualitative researchers a way to assess the trustworthiness of their data, their analysis and the inferences made. These are credibility, transferability, dependability and confirmability (Tashakkori and Teddlie, 1998: 90). Bassey (1999: 75) developed these criteria into a sequence which enables the researcher to check these issues, which is very similar to the sequence given by Tashakkori

and Teddlie. This sequence will be used to examine the trustworthiness of the data, analysis and inferences made in this research study.

Trustworthiness of the data collection

Bassey's three criteria for this stage of a research project are:

1. Has there been prolonged engagement with data sources?
2. Has there been persistent observation of emerging issues?
3. Have raw data been adequately checked with their sources? (Bassey, 1999: 75; cf. Tashakkori and Teddlie, 1998: 90)

According to Bassey, 'prolonged engagement' is about

... spending enough time on a case in order to be immersed in its issues, build the trust of those who provide data and try to avoid misleading ideas. (p76)

The pilot study and the main case study were all carried out in the school in which I worked for most of the time in question. Two of the classes considered were taught by me, the others were taught by two of my colleagues. I observed the classes taught by my colleagues during the Year 7 study, and observed two classroom discussions in the class who took part in the Year 8 study. I carried out all the interviews personally, and had frequent informal conversations with students and teachers involved, which enabled me to build trust and check my ideas with those participating. This immersion in the work going on also enabled me to decide on emerging themes, and then to check these with the participants. In particular, I was able to use the interviews after the teaching modules to probe participants' views and to check my understanding against theirs. In Phase II, where I did not have any direct contact with the students, I used the analytical framework established during Phase I.

Trustworthiness of the analysis and interpretation

Bassey's criteria for this stage of a research project are:

4. Has there been sufficient triangulation of raw data leading to analytical statements? ...
5. Has the working hypothesis, or evaluation, or emerging story been systematically tested against the analytical statements?
6. Has a critical friend thoroughly tried to challenge the findings? (Bassey, 1999: 75; cf. Tashakkori and Teddlie, 1998: 91)

Triangulation of the data occurred within stages and across the two phases. Evidence from questionnaires, classroom transcripts, classroom observations, and student and teacher interviews were compared with each other within each stage of the pilot study and the main case study. Classroom recordings and observations and student interviews were not conducted in Phase II, but evidence from the questionnaires was compared with data collected during Phase I. Analytical statements were made mainly on the basis of the classroom transcripts and the questionnaires from Phase I. These were then checked against the other data collected in Phase I: the interviews with the participants, students' written work, the classroom observations, and my own research journal. My teacher colleagues also observed lessons, and discussed what they saw with me. Further triangulation occurred between Phase I and II, and the analytical framework was reassessed in the light of the data collected in Phase II.

According to Bassey, analytical statements are "meaningful statements", which are "firmly based on the raw data", and which may also "suggest the need for more specific data to be collected" (p70). Initial analytical statements generated by the pilot study concerned the value of the graphic calculator model and teaching method to enable students to learn algebraic techniques, and to make cognitive changes in their understanding of how letters are used algebraically. These suggested the need to collect data which would give me the

opportunity to explore students' understanding of letters more deeply, and the data collection for the initial stage of the main case study, the Year 7 study, was planned on this basis. The Year 7 and 8 main case study then generated further analytical statements. Further data were collected in Phase II to check these, and to improve the generalisability of the statements. The analytical statements that emerged are:

- The graphic calculator model of a variable helps students in the early stages of learning algebra to develop a sound interpretation of letters.
- The graphic calculator method of learning algebra helps students in the early stages of learning algebra to understand basic algebraic syntax.
- The graphic calculator method of learning algebra helps students to make cognitive changes in their interpretation of algebraic syntax.
- The graphic calculator method of learning algebra helps students to remediate certain misconceptions in their understanding of letters.
- The combination of two students and a graphic calculator can constitute a ZPD in which both students are enabled to further their interpretation of letters and their understanding of algebraic syntax.

Throughout the analysis of Phase I, the emerging themes evolved, together with this analytical structure. These were tested against the data of Phase II, and the whole story was re-evaluated during the writing of this thesis. In addition, parts of this research have already been presented at conferences and seminars (Gage, 1999b, 2001, 2003a, b), and published in research and teachers' journals (Gage, 1999a, 2002a, b). This has allowed for critical challenge in addition to that provided by my supervisors.

Trustworthiness of the reporting

Bassey's criteria for this stage of a research project are:

7. Is the account of the research sufficiently detailed to give the reader confidence in the findings?
8. Does the case record provide an adequate audit trail? (Bassey, 1999: 75; cf. Tashakkori and Teddlie, 1998: 91f)

The account of the research is contained in detail in this thesis. All data and analyses are available should an audit be required; teaching materials and research instruments are contained in the Annexes to this thesis.

3.6.2 Generalisability of the conclusions of this research project

The generalisability of the findings of this research study is affected by a number of issues. These include the sampling, the nature of the data collection, and the analytic process. The participating schools and classes were selected on the basis of accessibility. The pilot study and main case study looked at the effect of using the graphic calculator in various classes chosen from a single school. This school was a girls' selective school, and so not a very representative environment for school children in general. The survey was more representative in that it included classes from four mixed non-selective schools. However, of the 12 classes included in the analysis of the survey, eight are from one school. These schools were self-selected, because there was a teacher at each of them willing to participate in the survey. Nevertheless, the students who participated in the survey did not choose to participate, and they covered a range of age groups and achievement levels (see Table 3 and Table 4, section 3.3.2). They attended four different schools in different parts of England, all co-educational and non-selective. This should permit generalisation of the findings to a wider population.

In Phase I, a good variety of different data types was collected. However, the coding of the questionnaires, from which much of the subsequent data analysis proceeded, required some personal judgement. Concluding that $6a + 2a = 8$ is algebraically incorrect was not a problem, but deciding a reason for the error was often more difficult. It could be that the student was simply ignoring the letters. Then again, the student might be substituting the value 1 for a and evaluating on this basis. In making judgements of this type, I was guided by students' responses to other questions. If the subsequent question was answered $12b - 2b = 10$, I would feel justified in inferring that the student was ignoring the letters. If the subsequent answer was $12b - 2b = 20$, I would infer that the student was substituting values, with $a = 1$, $b = 2$. However, there were occasions when some degree of uncertainty was unavoidable. I found that asking students in the survey how they had answered a specific question of the type $4a + 3b + 2a =$ was invaluable for reducing this uncertainty. This question also helped me to deduce that although, when specifically asked about their interpretation of letters, some students wrote that letters represented numbers, in fact they were working with the letters in some other way. Triangulation of the data during Phase I, and across Phases I and II, also helped with these judgements.

Another issue affecting generalisability is my choice of themes and analytical statements. Other researchers working with my data might well choose quite different themes and analytical statements. My personal biases have been at the root of my preference for the particular themes and statements on which I have chosen to work. Nevertheless, these themes were found through examination of the data, and although there may also be others which I ignored, those chosen are relevant and appropriate.

3.7 ETHICAL ISSUES

Ethical issues which needed to be addressed included obtaining informed consent from all those involved in any aspect of the research study. I also needed to consider whether the identities of those involved should be concealed or not. In Phase I, which involved students for whom I had a responsibility as a teacher and colleagues I was working with on a daily basis, it was also important that my work as a researcher did not impact upon my work as a teacher in any way that could be considered detrimental to my students or colleagues.

At all stages of the pilot study and main case study, I kept the Headteacher and my Head of Department informed about what I was doing and what my intentions were. I demonstrated to my Head of Department that the students' learning would not be compromised in any way by what I was doing. Letters were sent home to the parents of all students involved telling them about the research project and the teaching modules, and encouraging them to make contact with me if they had any questions at any point about any aspect of the study. The parents of all students taking a direct part in the data collection were asked for written consent for their daughters' participation in these activities. The students were told about what I was doing and its purpose, both as a research project and as a means of evaluating a new teaching approach. In Phase II, I asked the teachers I contacted to ensure that they asked permission as appropriate from their Headteachers and from the parents of students involved. To assist this, I sent a letter, which could be adapted for use as required, explaining what I was doing and inviting parents to contact me for further information if they wished.

Students taking part in any of the interviews or classroom data collection were all volunteers. They and their parents were asked for permission to use their words and work in any future publication in teaching or research journals or in this thesis. I decided that keeping the schools' and the students' identities hidden would prevent identification of individuals or any potential embarrassment to any person or institution in either phase of this study.

3.8 SUMMARY

In this chapter, the case has been made for a mixed methodology as a valid means of investigating research questions. This was then related to the specific investigation conducted during this research project. The research design was described in detail as a sequential design, in which a qualitative methodology was used in the first phase to enable rich detail to be gathered about the questions being studied. This was supplemented with a survey to allow generalisation to a wider population. Data collection and analysis was discussed, with attention paid to the trustworthiness of both these processes. The generalisability of the findings was considered, as were ethical issues which arose during the study.

In the following four chapters, the analysis of the data is considered in more detail, to present findings across the four major themes identified. These are:

- the role of the graphic calculator in mediating students' learning (Chapter 4);
- evidence for cognitive change in the students' interpretation of letters, and their ability to work with algebraic expressions (Chapter 5);

- evidence for progress in the students' understanding of letters and their ability to work with simple algebraic expressions (Chapter 6);
- the nature of students' misconceptions and evidence for the role of the graphic calculator in enabling students to remediate these (Chapter 7).

CHAPTER 4 THE GRAPHIC CALCULATOR: MEDIATING IN A LEARNING ENVIRONMENT

4.1 INTRODUCTION

In this chapter, the theorisation of the graphic calculator as a cognitive tool (section 4.2), is grounded in Vygotsky's theory of the mediation of tools (section 2.4.4). This is used to support the model of a variable which the lettered stores of the calculator embody. In section 4.3, the role of the graphic calculator in shaping students' higher mental processes is discussed (which was also considered in sections 2.4.4 and 2.5), and in section 4.4, its use as a tool for learning algebra is explored (continuing the theme of section 2.7). Section 4.5 contains a summary of the chapter.

The pre-eminence tools have in Vygotsky's work is a direct result of his grounding in Marxism. The Marxist analysis differs from much Western science however, which is based on an instrumentalist view of the notion of tool (Meira and Lerman, 2001), and which has its roots in positivism. Vygotsky worked with binary concepts, such as thought and language, learning and development, where the two poles are in a dialectic relationship to each other. In this dialectic, there is a tension where neither pole can exist without the other, and in which neither on its own is succeeded by some new synthesis. The interaction of the two *is* the synthesis. In Vygotsky's theory, tool use is internalised by us, with this internalisation becoming part of the resulting action. Meira and Lerman described this as "a result and a tool" with the "result' ... itself a tool" (p13). It is tool use in this sense that is explored in this chapter. The students' use of the graphic calculator and the constructs they form are both part of one conceptualisation.

Another metaphor used to tease out how the graphic calculator aids student understanding is Salomon's (1991: 186f) "cognitive prism" (section 2.5.2). This is also used to provide another perspective on the model of a variable provided by the graphic calculator. The model acts as a 'prism' through which students can pass questions and conjectures, which are confirmed or rejected.

It is argued in this thesis that the unit formed by a student pair and a graphic calculator has the potential to form a zone of proximal development, in which the students can develop their understanding (2.7.3). The view of the ZPD taken here agrees with that of Lerman (1998) and Meira and Lerman (2001), who conceptualise the ZPD as a symbolic space rather than some kind of force-field around a student. Two students discussing their ideas, and using the graphic calculator to verify or reject their conjectures, will learn more than either would have done alone or without the graphic calculator.

Using another binary concept of Vygotsky's, that of scientific and everyday concepts (section 2.4.4), the physical model provided by the calculator is discussed in relation to the abstract nature of the concept of a variable. Vygotsky held that scientific and everyday concepts need to interact if a true concept is to be formed. From this perspective, the calculator is viewed as a locus for this interaction of the abstract and the concrete to occur. Vygotsky held that thinking in concepts was impossible in the absence of verbalisation of thought (section 2.4.4), and the role of the graphic calculator in providing language and a focus for discussion is also explored.

The role of the graphic calculator in the four classroom studies reported in this thesis is described, together with teacher and student comment. Here, the effect of the calculator is differentiated from the contexts of its use. The underlying model, and the feedback of the calculator, are directly attributable to the calculator, and can be used in a variety of teaching situations. The teaching contexts are also discussed, however, as using the calculator with materials written specifically to support it provides a rich learning environment.

4.2 MEDIATION BY CULTURAL TOOLS

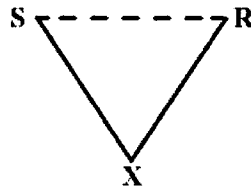
4.2.1 The nature of cultural tools

Wertsch's (1988) analysis of Vygotsky's work identified higher mental functioning mediated by socioculturally evolved tools and signs (Minick, 1987; Vygotsky, 1987), as the third major strand in Vygotsky's theory (cf. section 2.4.4). Indeed Leont'ev (1997), a younger colleague of Vygotsky's, said that Vygotsky began his analysis of mental processes with the analogy of the mediation of the labour process by tools. A quotation from Bacon, which summed this up for him, was:

Neither the naked hand nor the understanding left to itself can effect much. It is by instruments and aids that the work is done. (Bacon, 1620/1960:39, cited in Leont'ev, 1997: 17)

Where the behaviourists saw individuals reacting to the environment in elementary ways, with stimuli producing direct responses, Vygotsky envisaged a more complex interaction with the environment. A stimulus, S, would act through a mediating tool or sign of some kind, X, to produce a response, R:

Consequently, the simple stimulus-response process is replaced by a complex, mediated act, which we picture as:



(Vygotsky, 1978: 40)

Mediation can be provided by physical tools, or by psychological signs of which language is the most important.

Complex mental processes cannot be simply learned however, but have to be developed in the child (Vygotsky, 1987), and tools or technology, together with social interaction, can aid such development (Landsmann, 1991). The cultural background against which children develop in the early years of the twenty-first century is based on technology to a high degree, and children take for granted the presence of technology in most aspects of life. It therefore seems reasonable to base their learning in appropriate cognitive technologies.

The phrase “cognitive technologies” (section 2.5.1) was used by Heid (1997: 6) to express the idea of a microworld “in which students can express, develop and investigate mathematical ideas”. She linked this to other metaphors, like those of amplification and cognitive reorganisation, which are much used in the literature on cognitive technologies (Dörfler, 1993; Pea, 1985, 1987; Salomon, 1991). Tools are viewed as amplifiers of our physical and/or mental strength (Dörfler, 1993). Hence, the amplification metaphor is a way of explaining how technology can augment our powers, allowing us to carry out tasks more efficiently and in less time, but without changing the nature of those tasks. The

reorganisation metaphor recognises that technology can allow us to engage in new tasks, and to carry out mental operations which we could not have done on our own: the nature of the tasks we can accomplish is changed.

In section 2.5, various other metaphors for the action of cultural tools were also considered. These included Salomon's (1991: 186) "defining technologies" (2.5.2), which act by creating metaphors which come to serve as "cognitive prisms" (p186f) through which other phenomena are examined and defined. Salomon interpreted such a metaphor as a mental tool which can make the incomprehensible comprehensible (such as the universe in Oresme's use of the clock metaphor, section 2.5.2), and which help us to reorganise already acquired knowledge. Salomon also gave the example of the computer, which is used as a cognitive prism through which we gain an understanding of our own minds. It is argued in this thesis that the graphic calculator is another such cognitive prism, which serves to allow exploration of, and to make comprehensible, the concept of variable. However, we need to be aware that a feature of metaphors is to direct our attention to certain aspects of a phenomenon at the expense of other aspects, and that they may obscure as well as reveal.

There is some evidence from the four classroom studies reported in this thesis of the graphic calculator acting as an amplifier, enabling the students to concentrate on the concepts involved by taking away some of the routine work. Students were able to try a variety of values in an equation, for instance, using the calculator to test them reasonably quickly. In the pilot study, one pair of students checked five different answers to one equation, initially guessing, then attempting to work out their answer more systematically,

and finally using the “equation way”¹ – at which point they got the right answer. Without the graphic calculator, they would probably have just written down their first answer and left it at that. By the time the question was marked, with a time delay before they received this feedback, they would have forgotten what the question was about, and would simply have written in the correct answer without re-engaging with the question. There is rather more evidence of cognitive reorganisation in these classroom studies, in which students can be seen reorganising their knowledge, and acquiring new knowledge, through their use of the graphic calculator (Gage, 2002a, b). This is considered more fully in Chapter 5:

Evidence of cognitive change.

Changing the tools changes the knowledge

Focusing on technology draws attention to epistemology: for new technologies – *all* technologies – inevitably alter how knowledge is constructed and what it means to any individual. This is as true for the computer as it is for the pencil, but the newness of the computer forces our recognition of the fact. There is no such thing as unmediated description: knowledge acquired through new tools is new knowledge ... (Noss and Hoyles, 1996: 106)

The above passage was also quoted in section 2.2, but it encapsulates a thought worth considering again: Noss and Hoyles argued that the nature of the tools used in the acquisition of knowledge determines the structure of that knowledge. This suggests that the choice of appropriate tools and protocols shapes what is done, what is discovered, and how it is represented. Similarly, Salomon (1991) claimed that new technologies can lead people to raise new questions, to find new answers and to make new distinctions. This point has also been made by Noss:

¹ Their way of referring to the method they had been taught previously.

Learners can, in other words, say and do things with suitably-designed systems that they may be unable to say or do without them, and they can often do so in ways that are interestingly differently from conventional means. (2002: 48f)

It is argued in this thesis that the graphic calculator has the power to provide such a system for children meeting the algebraic use of letters for the first time, or struggling to make sense of earlier encounters. Although mathematical symbolic systems allow abstractions to be displayed, yet what they gain in universality they may lose in expressiveness (Noss, 2002). This is particularly true for the novice, endeavouring to make sense of a new representational system, but unable to see how such a system might allow expression of anything anyone might want to say. Students in the classroom studies² described in this thesis, responding to a question asking about the meaning of the letters in algebraic expressions, made this very clear³:

I think the *a* means above a number and *b* means below a number. [Year 6 student, school B]

An example of what you use? [Year 7 student, school D]

I think it means that it tells you that you're doing algebra. [Year 8 student, school A]

I think they are there to make it harder. [Year 8 student, school B]

Just fancy ways of making them look good. [Year 8 student, school C]

These students have varying previous exposure to algebra, yet it is clear that they have absolutely no idea what the letters are for, or why they should be using them. Algebra, if not completely meaningless, seems to be about doing "sums", then adding on "fancy" letters at the end, just to make it "harder". As Noss put it:

² Details of classroom studies can be found in section 4.4.3.

³ See also the responses to this question quoted at the beginning of this thesis, on p1.

... the compactness and elegance of mathematical expression does not necessarily make it equally functional for learning, and if learning is our prior goal, we would do well to think about new epistemological frameworks in which to embed the mathematics we wish our students to understand. (Noss, 2002: 60)

Such new epistemological frameworks could well include the use of appropriate cognitive technologies, such as graphic calculators. Students can learn:

... symbolic manipulations skills ... more quickly in areas such as introductory algebra... after [they] have developed conceptual understanding through the use of cognitive technologies... (Heid, 1997: 48).

Ultimately, use of the graphic calculator as a tool with which to introduce children to the concept of a variable needs to allow them to develop a rich understanding of a variable, and the skill to work with a variety of algebraic expressions, but the way that such understanding and skill are achieved may well be quite different from that of a learner in a more traditional teaching environment.

Effects of the tool and the context of its use

Two well-known meta-analyses of graphic calculator research differentiate between the use of the calculator, and the teaching approach in which it is used:

... it is uncertain if the [observed] shift to higher cognitive skills was a direct result of the use of technology or of the approaches used with it ... (Dunham and Dick, 1994: 14)

Similarly, Penglase and Arnold warned:

... studies which make claims regarding the *effects* of graphics calculator use must carefully distinguish between the *tool* and the *context* in which it is used, while those that purport to judge *effectiveness* must make explicit their assumptions concerning both the method and focus of assessment procedures ... (1996: 62)

The model of a variable provided by the graphic calculator can be considered as an effect of the calculator. The model is reliant on attributes of the calculator, and could be used with any number of teaching approaches. In the classroom studies described in this thesis,

the specifics of the teaching varied from one study to the next and from teacher to teacher, but the underlying model remained a constant. Another effect of the calculator is its ability to act as a tool for diagnosis and remediation through the instant feedback that it gives (Ávalos, 1996; Gage, 2002b). Again, this is not dependent upon the teaching approach. The Year 8 case study described in this thesis was taught by a teacher using her own resources. The graphic calculator model was used to underpin the concept of a variable, and students used the graphic calculator feedback to reassure them that they were making progress.

However, the effect of the calculator and of the context in which it is used cannot be separated entirely. The use of 'screensnaps', which are calculator screens for students to copy, gives a context particularly suited to exploiting the visual display of the graphic calculator. Their use also helps students by giving them a way to start a question. A student may feel completely helpless when faced by a question asking her/him to find a value for x that satisfies a given equation, not knowing where to start or how to write things down. That same student, if asked to copy a screen which requires finding a value of some unknown, can at least make a start. All they have to do to get going is to put a value into the appropriate calculator store, key in the required expression, and press the ENTER key. If it turns out to be the correct value, fine; if not, they can try again. Their efforts can be completely private if they want, with no one else aware that they do not really know what to do. This helps to take away some of the misery students often experience when faced with a page of algebra questions.

4.2.2 The graphic calculator model for a variable

Graham and Thomas (1998; 2000a) proposed a model of a variable using the graphic calculator (section 2.7.2). They suggested that the lettered stores of the calculator could be pictured (or, indeed, physically modelled) as 'boxes' into which numbers can be put, and that these could then be used to enable students to become familiar with elementary algebraic operations. The calculator allows the student to evaluate expressions and to keep an on-going check on whether their thinking is correct by operating on the letters in the same way as an algebraic variable. In this way, students are enabled to construct their own understanding of what the letters used in algebra mean. They do not have to learn to operate on them without any mental picture of what is happening; neither do they have to accept blindly what the teacher says.

Graham and Thomas' graphic calculator model is intended to help students begin to understand what the mysterious x used in algebra actually is. x or y , or any other algebraic variable, is seen to be a number, specifically the number in the X or Y store⁴ in the calculator. Figure 4 shows three screensnaps, which students are invited to copy on their own calculators. In order to do so, they have to decide on appropriate values to put into the lettered stores, in this case, the G , A and B stores.

⁴ The graphic calculator uses capital letters, rather than the lower case letters usual in algebra.

Figure 4: Three screensnaps

This leads to a number of questions which might be asked:

- What number do you have to store in G to copy the first screen?
- Will any other number work?
- What does $6G$ mean?
- What numbers can you put into A and B to copy the other screens?
- So does $A = 1$ and $B = 2$ in this instance?
- Is there more than one pair of numbers that will do?
- Do they have to be whole numbers?
- Do they have to be positive numbers?
- Do AB and BA always have the same value?
- What does AB mean?
- What does BA mean?
- How can you find out?

The graphic calculator can be a source of good questions for a class to consider, and it provides answers as well. If the value 5 is put into the G store, then the first screensnap can be copied exactly, and the students know immediately that they have found the correct value. If other values are put into the G store, these are found to be incorrect. Similarly, the questions raised by the expression AB can be discussed, conjectures made and tested on the graphic calculator, with immediate feedback as to their validity (Gage, 2002a, b).

Screensnaps like these allow the students to explore the meaning of an expression in which a number and a letter, or two letters, are placed side by side without any apparent operation being shown. In this way, they can discover that the convention in algebra is that the letters and numbers involved should be multiplied. Misconceptions, like that of always choosing $A = 1$ and $B = 2$, can be considered and found to be an inadequate way of conceptualising these letters: while it is true that A could sometimes contain the value of 1, and B that of 2, that would not work in the example given (Gage, 2001, 2002b).

Exploring the two expressions AB and BA more generally, students can find that they are equal whatever numbers are put into the A and B stores, and this can be seen to be the same if the variables are called A , B , X , Y , or any other letter. Teachers may feel that this will cause the students to have too great a reliance on the graphic calculator as the provider of mathematical authority, and that the calculator showing something to be true for a limited selection of values is far from being a proof and, of course, this is true. However, the fact that $AB = BA$ is always true is an axiom, part of the structure of the real number system, rather than a result susceptible to proof. The alternative is for the students to accept that it is so on the authority of the teacher. There are times when it is appropriate for teachers to make their students aware of the limits of the authority of the calculator, but here it is a case of whether students are asked to accept the authority of the calculator or that of the teacher⁵.

Working with the graphic calculator allows students to try out many different ideas, seeing immediately if their ideas are correct or not. For instance, students often want to simplify

⁵ This issue is discussed further in this section.

expressions like $x + 8$, feeling unhappy at the lack of closure. Using the graphic calculator, they can experiment with various such 'simplifications', discovering for themselves that these will not do, and thereby acquiring confidence in their ability to recognise a correct algebraic answer. In cases like this, it seems preferable for the student to be able to experiment until they are satisfied that $x + 8 \neq 8x$, for example, than just to accept from the teacher that it is not so. It is not enough to try examples with numbers, since for the student, there is no way of knowing whether what they are doing is correct or not, without some provider of authority. If $8x$ can mean 8 multiplied by x , maybe $x + 8$ can mean the same as $8x$? Why not, if it all seems just a game of arbitrary rules⁶?

The graphic calculator as a tool for diagnosis and remediation

A diagnostic tool in the educational context is one which identifies errors or misconceptions in a student's thinking. An example of the graphic calculator acting in this way is shown in this excerpt from the discussions of a pair of Year 9 students during the pilot study carried out for this research project:

Eleanor	Now on to question 3.	<i>This asked them to solve the equation $6x/4 + 9 = 4.5$.</i>
Kerry	OK, so ...	
Eleanor	This one seems quite a tricky one so we're just going to have to have a stab in the dark.	<i>These students had been using an appropriate method for solving previous equations, but appeared to be put off by the apparent complexity of this question.</i>
Kerry	Yes. OK, so we'll try ... 7.	<i>This looks like a complete guess.</i>
Eleanor	OK, 7. 7 STORE which is this key ⁷ , X,T, ENTER.	<i>They stored the value of 7 in the X store on the graphic calculator.</i>

⁶ It became clear during the analysis of the Year 6-8 survey, that many students interpreted $2a$ as $2 + a$, and bc as 8, given values of 2 for b and 6 for c .

⁷ The calculators the students were using had a special key, labelled X,T, for the variables X and T (used in parametric equations).

Kerry	OK, so now it's stored as 7. CLEAR screen. OK.	<i>Having cleared the screen, ...</i>
Both	6X divided by 4 plus 9	<i>... they started to key in the equation.</i>
Eleanor	... 2 nd function, MATH, ENTER, 4.5, ENTER.	<i>The key sequence 2nd function, MATH, ENTER gives a Boolean '=' sign. The calculator returns a 1 if the value in the X store is correct, but a 0 otherwise. Clearly, their calculator returned a 0 ...</i>
Kerry	No.	
Eleanor	No.	
Kerry	OK, so 7 ...	
Eleanor	...was not right.	<i>... telling them that 7 was not the correct choice.</i>
Kerry	Um ... so what are we going to try now?	
Eleanor	3. 3. Do you want to do it?	<i>They made another guess, this time that x is 3. Needless to say, this was also wrong.</i>

In this excerpt, the graphic calculator acted as a diagnostic tool, showing that these two girls had problems with equations of the type $ax/b + c = d$. Having tried several small, positive values they realised that perhaps the answer needed to be a negative number, but then, resorting to guesswork again, tried -24 as a “stab in the dark”, then -15 and -25. Kerry then said that would not be right, and that it would be 20 or 30. They settled on -29.5 instead, which was still wrong. At this point, Eleanor suggested that they “try the equation way” (that is, they worked out the answer rather than guessing it) and they very quickly calculated that x needed to be -3, which the graphic calculator confirmed as the correct answer.

This incident exemplifies the graphic calculator diagnosing a problem, then giving the students the opportunity to solve their problem for themselves. They did not need outside help, as they actually had all the skills they needed already. What they did need was feedback as they tried their guesses, helping them to realise that guessing was not an adequate method, so that they would revert to the method they had been taught. With its immediate feedback, the graphic calculator can provide support so that students are

empowered to solve their own problems. This is an example of the graphic calculator and the two students forming a ZPD (see sections 2.4.4 and 4.3) in which both students advance their understanding.

Hindrances to learning

It became clear from some of the transcripts of classroom discussions, however, that poor use of the graphic calculator can hinder students' learning. In the following extract from the Year 8 study, the students used the graphic calculator inappropriately leading them to confirm incorrect answers. These students were trying to solve the equation $2(p + 5) = 24$. This example also shows that at that point the students' thinking was procedural, rather than proceptual⁸.

Rebecca:	... so you must do the p plus the 5 first, and then add the 2. So p add 5 ... add the 5 and the 2 together ...	<i>Rebecca wanted to do the operation in brackets first, thinking procedurally, rather than seeing $(p + 5)$ as an entity, defined by the brackets. She also saw the 2 as added to, rather than multiplied by, the result of this calculation.</i>
Fran:	... 7 ...	<i>Because they did not see $(p + 5)$ as an entity multiplied by 2, they added the 5 and 2, giving them $7p = 24$.</i>
Rebecca:	... that's 7, and then 24 minus 7 is ...	<i>They then compounded this error by subtracting instead of dividing ...</i>
Fran:	... 17, so we guess that p is 17.	<i>... to obtain a completely erroneous answer.</i>

The graphic calculator should have helped them to at least realise something was wrong, but unfortunately the check they did simply repeated their mistakes. Instead of putting the

⁸ Proceptual thinking (section 2.6.5), a term coined by Tall and Thomas (1991), encapsulates the ability of the expert to view algebraic expressions flexibly, seeing the procedural aspect of an expression like $x + 1$ and identifying it as a mathematical object in its own right.

value of 17 in the P store, and then evaluating $2(p + 5)$, they used the graphic calculator to add 2 and 5, and then subtracted it from 24.

This is an example of calculator authority which needed to be challenged. The two girls clearly did not realise that what they were doing was completely wrong, and an inappropriate check meant that the graphic calculator feedback simply confirmed their error. Instances like this require the teacher to be aware of this kind of confusion, and to discuss the issues with the class. A similar question was discussed with the teacher concerned during her interview after the Year 7 case study a year earlier. On that occasion, she said that the students knew about brackets and where to find them on the graphic calculator, and so she could not see why there would be a problem. A year later, in her interview at the end of the Year 8 case study, commenting on a very similar equation to the one Rebecca and Fran were doing, it became clear that she had not been aware of the inadequacy of the checks some of the students were making. This is discussed in more detail in Chapter 5: *Evidence of cognitive change*.

4.3 THE GRAPHIC CALCULATOR: SHAPING HIGHER MENTAL PROCESSES, EXTENDING THE ZPD

In his *Prologue to the English Edition* of Vygotsky's *Thinking and Speech*, Bruner (1987) wrote that Vygotsky's theory of the zone of proximal development (or ZPD) was a "stunning concept" (p4). Late in his life⁹, Vygotsky saw that just looking at the current level of development of a child did not tell the whole story about that child's abilities. His

⁹ First mentioned in a lecture in 1933, some 15 months before he died.

example (also discussed in section 2.4.4) was that of two children who without help both perform a task at the level of an eight year old, but with help, one could perform at the nine year old level whereas the other could perform at the twelve year old level.

Vygotsky concluded from this that it was necessary to establish two levels of child development: what the child can do now, and what her/his potential is. The difference is the “zone of proximal development”. What is in a child’s ZPD today will be achieved tomorrow (Vygotsky, 1987: 211). If a problem or activity is within a child’s zone of proximal development, s/he will be able to solve that problem or perform that activity given help, although s/he would not be able to do it alone. An adult or more able peer provides a ‘scaffold’ which allows the child to reach a functionally more sophisticated level of performance. We saw an example of this in section 4.2.2, where the graphic calculator helped Eleanor and Kerry to solve an equation which they were finding difficult.

Lerman (1998) has argued that the zone of proximal development belongs to the classroom, rather than to the child, and that it:

... is created in the learning activity, which is a product of the task, the texts, the previous networks of experiences of the participants, the power relationships in the classroom, etc. (p71)

As has already been discussed in section 2.4.4, there has been criticism of conceptualisations of the ZPD as a force-field or physical space (Meira and Lerman, 2001; Newman and Holzman, 1993) which the teacher must find in order to teach successfully.

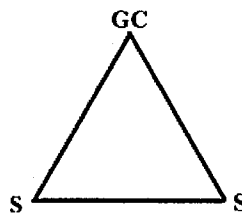
The revolutionary function of the ZPD is that it is the space, created in activities, in which the participants teach each other and learn from each other, where the dialectic of thinking and speech is manifested, and where the individual's meanings encounter social meanings (sense) and purposes. (Meira and Lerman, 2001: 4).

Indeed:

... pairs of students can create their own zone of proximal development if they are motivated, taught how to share ways of working, have an appropriate personal relationship, and/or other factors. (p72)

It is argued in this thesis that the unit of a pair of students and a graphic calculator can create a ZPD for the students, if their working relationship satisfies these conditions.

Figure 5: Model of ZPD created by a pair of students and a graphic calculator



This unit of student-graphic calculator-student provides another metaphor for exploring how the graphic calculator enables students to learn, supporting them together as they explore the rules and conventions of algebra.

... learning can be facilitated by providing help in developing an appropriate notation and conceptual framework for a new or complex domain, allowing the learner to explore that domain extensively. (Noss and Hoyles, 1996: 107)

A pair of students plus the graphic calculator provides the scaffolding needed to support the students in reaching a higher level of understanding. The graphic calculator provides the notation and conceptual framework for the new domain of algebra, but on its own would be insufficient, since it can only give feedback, it cannot suggest another approach. The student partner alone would not be adequate, since s/he would not know if a different approach was any better. The combination of two students and a graphic calculator, however, provides an environment in which cognitive development can occur, with the calculator acting as the tool which mediates that development.

4.3.1 Scientific and everyday concepts

Vygotsky hypothesised that children learnt both *everyday*, or *spontaneous*, concepts and *scientific* concepts during childhood and adolescence (section 2.4.4). Scientific concepts are those which are taught to a child, probably in school, and which derive from formal learning. These he visualised as descending from the abstract to the concrete, where they meet up with the spontaneous, everyday concepts which originate in the child's everyday experience. Luria, in the *Afterword* to the Russian edition of *Thinking and Speech* (Vygotsky, 1987), argued:

Everyday concepts are well known to the child. He knows what a house is, what a dog is, and what a brother is. He uses these concepts effectively, but cannot provide verbal definitions for them. Everyday concepts do not enter the child's conscious practice in a direct way. The opposite features characterize scientific concepts. Scientific concepts are introduced by the teacher through verbal means even before the pupil has any concrete experience with what stands behind them. As a consequence, the pupil can easily formulate the scientific concept verbally. This does not mean, however, that he can use the concept fluently. (p366).

Vygotsky was very critical of the type of teaching which expected children to learn from direct instruction alone, which is often all they receive when learning algebra, stating that this is "a mindless learning of words, an empty verbalism" (Vygotsky, 1987: 170). The teaching needs to be seen as a beginning, rather than an end, with much work still to be done on the child's part:

... scientific concepts are not simply acquired or memorized by the child and assimilated by his memory but arise and are formed through an extraordinary effort of his own thought. (pp176f, original italics)

In order for children to learn new concepts successfully, there needs to be teaching which will introduce them to the abstract aspects of the concept, and there needs to be practical experience which will give them a concrete grounding in using that concept. The abstract and concrete should not be viewed as "bipolar opposites" (Noss and Hoyles, 1996: 45)

however, but rather as a dialectic. Meaning is “reshaped in the interplay between ‘abstract’ and concrete’ activities” (p45). Another way of expressing this is that:

... concreteness is not a property of an object but rather a *property of a person’s relationship to an object* ... The more connections we make between an object and other objects, the more concrete it becomes for us. ... This view will lead us to allow objects not mediated by the senses, objects which are usually considered abstract – such as mathematical objects – to be concrete ... (Wilensky, 1991: 198f, original italics).

The graphic calculator model brings together scientific and everyday concepts for a variable. The everyday concept of a number in a box or store is brought together with the algebraic conventions, expressions and equations which students meet in school, thus enabling them to move from the abstract to the concrete and from the concrete to the abstract. Working with the calculator also enables students to make connections between the physical model of a variable as a number in a lettered store and the abstract algebraic concept. When classes are working with the graphic calculator model of a variable, it is therefore very important that the teacher uses opportunities to draw out the significance of what the children are discovering. This allows students to work in their ZPD, which has the potential for a:

... fundamental reconstruction of the child’s reflection of reality ... the creation of new psychological formations of a kind that the child’s spontaneous development could never have achieved (Luria, 1987: 367).

Dörfler (1993) similarly asserted that cognitive processes can often be

... successfully guided and organized by concrete representations, images or models of the given situation. The thinking process then consists essentially of transformations and manipulations of these ... models. (p166)

He further argued that the difference between novices and experts is the range and adequacy of the mental models they have available to them. Construction of such models is not simple, nor is it automatic, with many students failing to produce effective models for themselves. In providing novices with an easily understood model for a variable, the

graphic calculator is helping them to do mathematics more as an expert would. The support given by the graphic calculator is not that of a 'temporary crutch', but a full part of the cognitive construct built up by the student, similar to the function of diagrams and physical models – a tool-in-use becoming a "tool-and-result" (Meira and Lerman, 2001: 11).

It is an important aspect of models used to enable learning that students should be able to move on from them. This is emphasised by Noss and Hoyles (1996: 107):

Learner participation is gradually increased – according to the needs and learning pace of the individual – and the support is gradually *faded*. (Original italics)

The fact that the graphic calculator initially enables the student to create a cognitive construct for a variable does not mean it always has to be physically present to be useful. Vygotsky noted that children learn from their teachers initially by imitation and collaboration, then by using their awareness of the "adult's help, invisibly present" (Vygotsky, 1986: 191), so that eventually they can operate independently.

4.3.2 Language with which to think

Thinking in concepts is not possible in the absence of verbal thinking. (Vygotsky, 1978: 131)

The central tenet is that verbalisation helps students to own their knowledge, to ask realistic questions and to make mathematical structures and relationships explicit. Discussion can ... assist in focusing awareness between the tool use of mathematics and the appropriation of the relevant structures and relationships. (Noss and Hoyles, 1996: 141)

These two quotations both emphasise the need for students to have language with which to think (section 2.4.4). Vygotsky made the point repeatedly that higher mental functioning requires words, whether in the context of spoken language or of internal thought.

Similarly, Noss and Hoyles claimed that learners can think and speak about abstract,

mathematical ideas, if they have a language with which to do this, and that this language can be derived from the tools and notation used. It is argued here that the graphic calculator can provide the language of algebra, by providing a tool with which to work, notation to express the relationships between variables, and words with which to describe these relationships (cf. Ávalos, 1996).

The model of a variable as a 'box' or 'store' containing a number can be used to explain the use of letters in algebra. This is then exemplified with the use of screensnaps, so that students become used to putting numbers into the lettered stores of the calculator. Each time they do this, it underlines that the letters are placeholders for numbers, giving students a way of explaining what letters are in a way that makes sense. It also gives the questions a meaning. When a student is asked to copy a screensnap, there is an action for them to perform: they put a number into the given lettered store, and see if the screen they get when they press the ENTER key is the same as the screen given. The following brief excerpt from the discussions held by a pair of Year 7 students shows how such internalisation of the concept of a variable occurs in the context of the graphic calculator:

- | | | |
|--------|---|--|
| Sam: | It's 2 times B , 7 is in B , so it's 14. | <i>This question asked them to evaluate $2 \times B$, given that $B = 7$. Note the use of language here: "7 is <u>in</u> B". This language is provided by the graphic calculator model.</i> |
| Chloe: | It is 14. Next one is $2B$. So it's 2 times 7, 14. | <i>The next question asked them to evaluate $2B$.</i> |
| Sam: | $2B$ is 2 times B . | <i>Remarks like this help the students to internalise the fact that $2B = 2 \times B$.</i> |

In the Year 6-8 survey questionnaires, five students answered a question about their interpretation of letters by using language directly provided by the graphic calculator on

the post-questionnaires, having failed to answer the question satisfactorily on the pre-questionnaire. These are the responses of two Year 8 students from school A:

I think it means that it tells you you're doing algebra. [*First student, pre-questionnaire*]

They stand for the numbers which you stored in the graphic calculator. [*Immediate post-questionnaire*]

I think that's what they are stored in. [*Delayed post-questionnaire*]

[Question left blank] [*Second student, pre-questionnaire*]

I think a and c means what number is stored in A and C . [*Immediate post-questionnaire*]

I think a is a number like 5 and b would be like 10. [*Delayed post-questionnaire*]

All five students made good gains in their total scores on the algebraic questions between the pre-questionnaire and the delayed post-questionnaire: from 17% to 27%, 8% to 27%, 25% to 45%, 8% to 27%, and from 42% to 73% respectively. I would argue that the calculator gave them the language with which to conceptualise these problems, thus enabling them to tackle them more successfully.

A focus for reflective discussion

It has been claimed that working with technology tends to increase the proportion of group work done in classes (e.g. Heid, 1997). More recently, Hennessy, *et al* (2003) have carried out detailed classroom research, which suggests that much of this group work is not really collaborative, and that opportunities for genuine collaboration need to be specifically provided by the teacher, rather than this being a necessary outcome of using ICT. When such collaboration does take place, however, there is opportunity for discussion to take place, both between small groups of students with or without the teacher, and between the

whole class and the teacher. Such discussion has the effect of making students more conscious of their views through re-presenting them to others:

In the effort to communicate, the speaker has to strive to frame his or her thoughts in language which conveys meaning, to try to see another point of view and develop more flexible approaches to strategies and solutions. (Noss and Hoyles, 1996: 142)

Disagreement, which may then ensue, gives students the opportunity to re-evaluate their initial thoughts. Discussion is therefore a form of scaffolding and contributes to the ZPD: collaboratively students can achieve more than they could alone. In the Phase I case studies, the students were encouraged to work in pairs so that reflective discussion would be facilitated, and real collaboration could occur. This approach was also suggested to the teachers at schools A, B, C and D, as a way of enabling reflective discussion between students. Such discussion needs to be focused if it is to be an effective learning instrument however, and the graphic calculator provides such a focus (Graham, 1998).

In a questionnaire completed at the end of the pilot study, those students who had recorded their conversations in the classroom were asked if they thought this had made a difference to their work. Nine students answered this question, and of them six felt it had made a significant difference, two did not think it had made much difference and one felt it had made no difference. It is interesting to note that some of the students were aware that they were working more slowly than normal¹⁰, and that by saying what they were doing aloud, they were taking more notice of what they were doing. Explaining for the benefit of the tape¹¹ also became explaining for their own benefit. Remarks included:

As you said what you were doing it kind of stuck in your brain.

¹⁰ This issue is discussed by Hennessy *et al* (2003). Teachers in their studies found that the pace of lessons was often less where ICT was used.

¹¹ The tape recorder forced the students to articulate their thoughts, but the graphic calculator provided the forum for those thoughts. The two effects are not the same.

We had to explain what we were doing all the time whereas usually we'd just do the work without discussing it so much.

It made us explain our work and not have to rush ahead. We also understood our work easily.

I think it made a difference because we had to concentrate harder and not gossip as much!

It slowed us down quite a bit, having to explain everything as we went.

4.4 THE GRAPHIC CALCULATOR AS A TOOL FOR LEARNING ALGEBRA

4.4.1 Problems with traditional teaching approaches

... a focus on procedures alone, without conceptual linkages between them, leads to increasing cognitive stress as the individual learns more and more disconnected pieces ... (Tall, 2000: 37)

The classroom work described in this thesis originally arose from a desire to offer students something more meaningful to them than the normal paper and pencil approach for teaching the early stages of algebra. Many students fail to grasp much about the nature of a variable despite years of algebra lessons, indicating that traditional methods do not succeed for all students (section 2.6). Indeed such difficulties are a commonplace among secondary school mathematics teachers and researchers.

Kaput (2000) claimed that “in contrast with the arithmetic system, *the algebra system was built by and for a small and specialized intellectual elite*” (p5, original italics), regardless of the difficulties most learners would experience in trying to grasp its principles.

The effect of these learnability factors did not really become felt until the latter part of the twentieth century when education systems around the world began to attempt to teach algebra to the general population. Prior to the middle of the twentieth century, the algebra literacy community was quite small, quite analogous to the small literacy communities of the specialists associated with early writing. (Kaput, 2000: 6)

Kaput went on to say that the invention of the alphabet had revolutionised the process of learning to read, so that it became accessible to the population at large, and that a comparable invention was now needed for algebra. He suggested that the graphic calculator was a suitable instrument, which was practical to use in the school classroom, and would facilitate students' efforts to learn algebra. Changing the medium of instruction would allow symbolic algebra to become knowable and learnable. Kaput was discussing the use of CAS¹² systems on graphic calculators, and the students he worked with were studying calculus. However, the focus of this thesis is to argue that the use of graphic calculators can aid much younger students in their learning of algebra.

4.4.2 Use of the graphic calculator in the case studies

Various types of use of the graphic calculator by students have been suggested. These include using them for calculations, for data collection and analysis, for visualisation and for checking (Doerr and Zangor, 2000). Other uses have also been observed, including exploration beyond the immediate topic being taught (Simmt, 1997). Some of these uses were also significant in the classroom work described here, especially checking and exploration.

In all the studies described in this thesis, the basic model was that of the graphic calculator stores providing an instantiation of an algebraic variable. The calculator's stores are labelled with letters; numbers can be put into the stores, and the calculator can then be used to display algebraic operations on the lettered stores and the results of these. The key

¹² Computer Algebra Systems, such as DERIVE. These are not the focus of any of the work described in this thesis.

sequences required follow the algebraic expressions exactly, and the display looks the same as an algebraic expression, apart from the use of upper case rather than lower case letters. In order to exploit this model, the students used the calculators as tools for investigation and exploration, and to provide feedback on their conjectures (Gage, 2002a, b). The screensnaps also provided visual confirmation of how a given operation on a variable should be expressed, a great help to students in the initial stages of learning algebra.

The students' work was based on the constructivist approach throughout: it was intended that the students' actions with the graphic calculator would help them to contextualise their understanding of letters used in the algebraic context. Knowing how to perform certain rituals in mathematics is not the same as understanding; the constructivist approach to teaching and learning views exploration and reflective enquiry as more useful than the passive acceptance of what the teacher says (Richards, 1991; Steedman, 1991). It was therefore intended that the use of the graphic calculator, and the materials provided, would enable students to develop their understanding first, and to allow the learning of skills to come from this or to be returned to later.

Discussion between students was viewed as a vital part of the learning process, both between pairs of students working together on the screensnaps and other questions, or between small and large groups of students and the teacher. The purpose of such discussion was for students to make conjectures about the screensnaps: to verbalise their ideas, to compare them with those of others, and to use the graphic calculator to test them. The teacher's role was to probe misunderstandings and ask questions that would make

students reconsider, to bring good questions to a wider audience, and generally to open up discussion.

4.4.3 The classroom studies

Details of the schools and classes participating in the classroom studies are given in section 3.3. The case studies were all carried out in school *G*, a girls' grammar school, at which I taught until Christmas 2000. A pilot study for this research was carried out in the summer of 1999 (Gage, 1999b), followed by the two stages of the main case study, in the autumns of 2000 and 2001 (Gage, 2002b). The pilot study involved one class of Year 9 students taught by me. The initial phase of the main case study was conducted with half a year group (three classes) of Year 7 students, taught by two of my colleagues and myself (Gage, 2001, 2002b). The following year, one of these classes, by then in Year 8 and taught by one of my colleagues, was followed up.

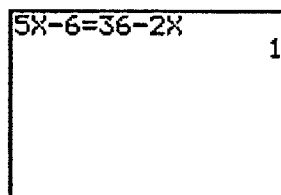
Year 9 pilot study, 1999

The classroom element of my graphic calculator research started in April and May of 1999 with Sally (section 1.2), a 14 year old girl who needed additional help, if she was to do as well as her peer group in the end of year standard assessment tests (Key Stage 3 SATs). Her knowledge of algebra in particular was limited and unreliable: when asked what $2x - x$ equalled, she answered very firmly: "2!" (Gage, 2002a). Using the idea of putting numbers into stores, then carrying out simple operations with the graphic calculator, she began to understand how letters are used in algebra, and to have some success with algebraic questions.

Working with Sally during lunch breaks then developed into a pilot study with her class, which was carried out during June and July of 1999. Sally was well below the average level of mathematical attainment of her year in that particular school. She was the only student with a Level 5 pass¹³ in the SATs exams that May, while some of her peers gained Level 8 passes and the rest gained Level 6 or 7 passes. For most of these students, the problem was not to understand basic use of letters, but to develop their skills on more challenging questions. However, other students were not so confident. I hoped that using the graphic calculator would stretch those for whom this was appropriate, and provide a means of building up a sounder conceptual basis for others. The topic chosen was that of solving linear equations and simplifying algebraic expressions of varying degrees of difficulty, since a number of the equations required simplifying before they could be solved. Easier questions were succeeded by more challenging ones so that everyone could work at an appropriate level.

The main teaching method was to give the students screensnaps (calculator screens) to copy. The following screensnap is an example from the pilot study¹⁴:

Figure 6: Screensnap illustrating method used in Year 9 pilot study


$$5x-6=36-2x$$

1

¹³ Although this was the target grade nationally.

¹⁴ This example shows the magnitude of Sally's problem: she was convinced that $2x - x = 2$, but she was expected to be able to solve equations of this complexity. I am convinced that many students are in this position, of being expected to work on problems which require considerably more conceptual understanding than they possess.

To copy the screen in Figure 6, the students needed to calculate the value of x which would satisfy the equation. They would then put this value into the X store, and key in the equation. If the value in the X store was correct a 1 would be returned (as in the left screensnap in Figure 7), but if the value was incorrect a 0 would be returned (as in the right screensnap in Figure 7).

Figure 7: Screensnaps illustrating use of graphic calculator in the pilot study

6→X	6
5X-6=36-2X	1
7→X	7
5X-6=36-2X	0

In addition to equations set in this way, others were given in the more usual way, but the students were still expected to check their answers with the graphic calculator.

In the simplification exercises, students were given typical algebraic expressions to simplify, for example:

$$2a + 3b - a - 4b$$

$$36a \times 2b$$

The screensnaps in Figure 8 show the first example simplified correctly (the calculator returned a value of 1), and the second simplified incorrectly (the calculator returned a value of 0). Students were told to put any values they liked into the A and B stores for

these exercises, although it was pointed out that it would be a good idea to avoid using 0, 1 and 2¹⁵.

Figure 8: Screensnaps showing how simplifications of expressions were checked in the pilot study



A major benefit of the graphic calculator was that it gave prompt feedback to the students: they knew immediately if their answer was correct or not, and could work on a question until they were satisfied. This is in contrast to much classroom practice, where students answer questions, writing answers in their books, then check them at the end of the lesson or wait until the book has been marked by the teacher. Often by the time the students find out if their answers are right or wrong, they have forgotten what the question was about and what their thinking was at the time, so there is no chance for them to change the way they thought. A correction may be written in, but this is a mechanical exercise, compared with the opportunity to rethink a question at the time (Gage, 2002b).

This excerpt from a transcript of a classroom exchange shows how the graphic calculator was used during this case study, and in particular, how it enabled the students to correct their approach to a problem:

¹⁵ Using 0, 1 or 2 could make expressions equal which would not otherwise be the same, e.g. $36a \times 2b = 38ab$ is correct if either a or $b = 0$; $p^2 = 2p$ if $p = 2$ (an example discussed at length in Chapter 5: *Evidence of cognitive change*).

- Gemma: Number 8 is $5X$ take 6 equals 36 take $2X$. $5x - 6 = 36 - 2x$.
- Lauren: So here we could take ...
- Gemma: ... take the $2X$'s ... *They decided to take away the $2x$ term, forgetting that the '-' sign is attached to it - a very common mistake ...*
- Lauren: ... take $2X$ and we would get $3X$ minus 6 equals 36. ... giving $3x - 6 = 36$.
- Gemma: Yeah.
- Lauren: ... and 36 ...
- Gemma: ... is it plus 6? 36 plus 6 equals 42 divided by 3 equals 14. X equals 14. Right then. *They reversed the operations to get x from $3x - 6 = 36$...*
- Lauren: Oh, yes, X equals 14. Check on the calculator. ... which gave them 14.
- Gemma: Yeah.
- Laura: So we do 14 STORE X , ENTER, then you do $5X$ take away 6 then 2nd TEST, ENTER equals 36 take away $2X$, ENTER ... we got it wrong. *They entered 14 into the X store, then input the equation. 2nd TEST, ENTER gives them the Boolean '=' sign. However they found that 14 was not the correct answer.*
- Gemma: We got it wrong? Um ... Would it be ... *After some discussion, and trying it the same way, again unsuccessfully ...*
- Gemma: Yes, so if we add $2X$ we'll get ... *... Gemma identified the problem.*
- Lauren: That'll equal $7X$... $5x - 6 = 36 - 2x$
 $7x - 6 = 36$
- Gemma: Yeah, take 6 ...
- Lauren: ... equals 36 ...
- Gemma: Yeah.
- Lauren: ... and now we do X times 7 minus 6 is 36, then 36 plus 6 which equals 42, 42 divided by 7 is 6, so X ... *When they repeated their calculation, they got the right answer of 6.*
- Gemma: Yeah.
- Lauren: ... so X equals 6.
- Gemma: Yeah.
- Lauren: Are we going to check it on the graphical calculator? 6 ENTER, then we do $5X$ minus 6, 2nd TEST, ENTER, 36 minus $2X$. Yes! We got it ... we got it right! So X equals 6. *They checked this answer with the graphic calculator ... and saw that they had then answered the question correctly.*

Year 7 study, 2000

The next study planned was with Year 7 students (aged 11-12 years) in their first term of secondary education. The pilot study had proved effective as a means of teaching Year 9 students about more complicated equations and expressions, but some of them had found using the Boolean '=' sign unduly complicated, and the data collected had not proved a particularly effective way of probing students' understanding of a variable. Three classes took part in the Year 7 case study (half a year group), with two additional teachers using the graphic calculator model. It was intended that this would give greater breadth to the findings (Gage, 2002b).

The graphic calculator model was used to introduce students to using letters. Many of them had done some work with letters in their primary schools, but for some it was their first introduction to algebra. The National Curriculum in force in the UK at the time required that students in the 11-14 age range were taught how to use symbols in equations and expressions, progressing from using a letter as a definite unknown to using a variable in an identity or function. It was intended in this study to explore students' initial exposure to expressions, what meaning they gave to the letters used in these expressions and the types of misconceptions that occurred in their thinking, both before and after the teaching module. The teaching approach encouraged constructivist learning, relying heavily on the calculator's provision of instant feedback, so that correct ideas were strengthened and misconceptions quickly challenged. Reflective discussion between pairs of students, and between the whole class and the teacher was also a feature of this study.

Again screensnaps were used, this time to allow students to investigate how letters are used in algebraic expressions, and to do simple operations on the letters. The students started by

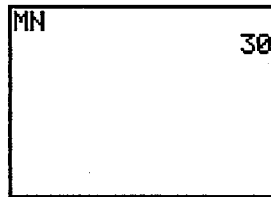
learning how to put a number into a store and to evaluate simple expressions (as illustrated in the left and middle screensnaps in Figure 9). Then they went on to predict values to put into stores to produce given screensnaps (as in the screensnap on the right in Figure 9).

Figure 9: Screensnaps illustrating use of the graphic calculator in the Year 7 case study

3→F		5→A		A+1	
F+1	3	2A	5	B-2	4
F-2	4	6A	10		3
	1		30		

Once the teachers were sure that their students knew how to produce screensnaps like these, the students worked together in pairs copying further screensnaps which took them through a range of algebraic ideas and conventions. Screensnaps were used throughout this study, with students using the graphic calculator to check their ideas. Common misconceptions which needed to be challenged were that $A = 1$, $B = 2$, so $AB = 12$ or 2; or if $AB = 16$, then $A = 1$, $B = 6$. Again, instant feedback was an important aspect, as was the remediation of misconceptions about letters, and how they are used in algebra (Gage, 2001, 2002b).

The two students, whose discussion is quoted below, started their first lesson thinking that if $AB = 16$, then $A = 1$, and $B = 6$. By this point in the lesson (less than an hour later), they had corrected this idea, and Fran was starting to express ideas about the range of numbers which could be used for a screensnap like this. This excerpt shows the power of the graphic calculator to help students sort out these ideas. Figure 10 shows the screensnap they were copying:

Figure 10: Screensnap used in Year 7 case study

- | | | |
|----------|---|---|
| Rebecca: | We've got another one. MN , which is M times N , equals 30. 6 times 5, because 6 times 5 is 30. | <i>The recognition that MN is a product shows a considerable step forward for these two students, who, less than an hour earlier, thought the letters were the digits of a two-digit number.</i> |
| Fran: | But you see, it could be anything. It could be one is 10 and one is 3. | <i>It is this remark that is significant however. It shows the beginning of the realisation that letters are generalised numbers, rather than specific numbers.</i> |

Year 8 study, 2001

My intention initially was to follow up all of the students who had taken part in the Year 7 study a year later. However by then, I was no longer teaching at the school, and one of the other classes had also had a change of teacher. I decided therefore to follow up the one class who had kept the same teacher, and to take a 'snapshot' of the students' thinking a year after their initial introduction to algebra with the graphic calculator. My focus was on how the students' thinking had progressed since the previous year, rather than on the classroom process or how the calculator was used.

The teaching method in this study did not concentrate so directly on the graphic calculator. Students had graphic calculators available at all times to use as they saw fit, but apart from a few screensnaps to remind them of the basic method, the teacher did not direct students on how to use the calculators. The focus was on the calculator giving feedback, so that students would be able to monitor their progress, rather than on the calculator providing an

environment for discovery and exploration. Rather than using materials specially prepared for the graphic calculator, as had been the case in both the previous studies, this time the teacher used resources of her own, using the graphic calculator to support the underlying model of a variable, rather than as a specific tool for exploring ideas about variables. The topic was on the use of the 'balance' method to solve linear equations, together with some work on graphs of linear equations.

Year 6-8 survey, 2002

By the end of 2001, I had collected data in a girls' grammar school with students aged 11 to 14 (school years 7, 8 and 9), from classes taught by three different teachers. However, there were clear limitations on using these data to draw conclusions which could be applied to students of this age more generally: only girls were represented, and they all came from one selective school. The next stage, therefore, was to find other schools which would use the graphic calculator model for a variable, would use materials specifically prepared to allow exploration of how letters are used and interpreted, and would allow the collection of data from their classes. The four schools which eventually took part were all mixed, non-selective schools, one a 9-13 school in East Anglia (school *B*), two 11-16 schools on the south coast (schools *A* and *C*), and one 11-18 school in the north-east (school *D*). The students who participated were in school years 6, 7 and 8, that is, aged 10 to 13¹⁶.

Again, screensnaps figured heavily in the teaching method¹⁷. After the same kind of start to the sessions as described in the Year 7 study above, students were given questions like

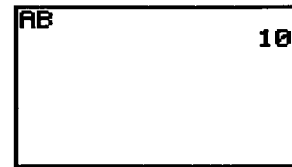
¹⁶ Full details of the schools and the students participating can be found in section 3.3.

¹⁷ The worksheets, teaching notes and general instructions can be seen in Annex I.

this to give them the opportunity to discuss aspects of how operations on letters should be interpreted:

Can you make this screensnap in three different ways? Write down the numbers you use.

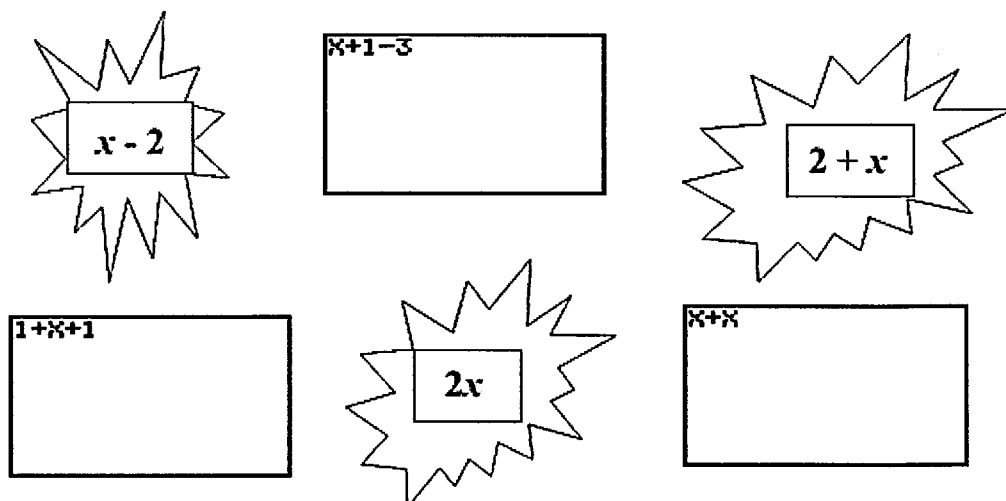
What would the screensnap for BA look like? Draw it.



Questions like this were followed by harder screensnaps, using large numbers, negative numbers or non-integer numbers, and more operations in each question.

Students were then asked to match equivalent expressions. The six expressions in Figure 11 formed one such set to be matched.

Figure 11: Screensnaps and stars



Questions like this gave students the opportunity to investigate sets of expressions, to find out which of the expressions are the same and which are not. Here the graphic calculator is acting as a tool for investigation and exploration, giving immediate feedback, so that conjectures can be made and tried out. This approach can be used to explore specific

misconceptions, for instance, that $2x = 2 + x$, a misconception held by a surprisingly large number of students (see Chapter 7: *Misconceptions*).

4.4.4 Participants' views on the graphic calculator

Teachers' views on the graphic calculator model

In interviews conducted at the end of the school year with my two colleagues from school G who had participated in the Year 7 case study, both said they had been very pleased at how well the students' understanding of letters and their competence with algebraic expressions had been maintained during that year. They felt that the students had made much better progress with other aspects of algebra, such as constructing and evaluating simple formulae, than would normally have been the case. They also felt the students' motivation to study algebra had not fallen as much as they had expected on the basis of their previous experience.

In the Year 6-8 case study, one teacher from each school completed a questionnaire which included questions about how helpful the graphic calculator had been. All four said they had found the graphic calculator model useful and that they would be prepared to use it in the future. They all felt that their students had coped well with using the graphic calculators, finding them a help rather than a hindrance. The teacher from school C wrote: "I had one moment where a very weak girl explained clearly to me what $5a$ meant. It was quite brilliant." She also said that she had already successfully used it with "a weak Year 11 group", and that she intended using this model in the future:

Ideas will be written into Y7 scheme of work for September. Rest of department have had a go – very successful. Even shared ideas with another school who were equally enthusiastic.

The teachers from the other schools expressed similar feelings, and had recommended this model to others in their departments.

The teachers in this study claimed their students' understanding of letters had progressed through using the graphic calculators. The teacher from school A said of his "more able" group that they "[n]ow have a firm model of a variable. Needs to be built up further, but progress!" His "less able" group showed "[s]ome improvement. May not remember or be able to work independently, but can grasp the concept when led." The teacher at school D commented that her students' understanding was "[m]uch better, ideas already formed have been corrected. More confident/less afraid of algebra."

The teacher from school A sounded a note of caution, however, which relates to the authority of the calculator:

I'm quite taken with this approach, but I've had one slight doubt for a while when we suggest that $3A$ means $3 \times A$, because that's what the calculator does. I'm not sure that's a good thing! 'It must be that, because the calculator says so.' However, the model is so useful, I've decided to 'teach round' this issue.

He was concerned that students would learn to accept an answer "because the calculator" says so. This can be a hindrance to learning, as discussed earlier in this chapter (section 4.2.2), where two students confirmed a wrong answer because they were using the calculator incorrectly. The calculator cannot remedy errors caused by using the wrong key sequence, or by putting unsuitable numbers into the stores. The deficiency is not in the authority of the calculator or the model, however, but in students' lack of awareness of the need to use the calculator correctly, and of the need to follow up problems, rather than just assuming that if there is a discrepancy between a calculator answer and an expected

answer, that the calculator is bound to be right. As the teacher at school A added, this is an issue that teachers need to be aware of, and to discuss with their students.

Students' views on using the graphic calculator

Very few of the students had any problems with the graphic calculator, once they had practised what to do a few times. The only students commenting to any extent on difficulties with, or dislike of, the graphic calculator were a few from the pilot study. Comments made by some of them about it being a nuisance to have to use a menu to access the Boolean '=' sign were the main reason for simplifying the use of the calculator between the pilot study and the other studies. In the later studies, all students really had to know how to do was to put a number in a lettered store, then copy the content of a screensnap, equation or expression.

One particular student in the pilot study really disliked using the calculator, but this degree of dislike was not found in the later studies, or even among other students in her class.

When interviewed, she said it was fiddly, and she and her partner begrudged the time it took to do a calculator check:

Interviewer: How do you think you are getting on using the graphic calculators?

Holly: I never ... They help just to, like, check things, if you've done it right, like those things we did yesterday, but I never really ...

Carly: I think they do help, it's just ...

Holly: ... they seem really complicated.

Carly: Yeah, you have to do so many things each time you do a question, and you just wonder if it is worth the bother.

Holly: It's too long.

Nevertheless these two did experience success with the calculator, finding that it helped them to get questions right.

Other students in the pilot study commented:

... and then we just check, and it's good to see if it's right ...

... it tells you if you're getting on, if you're like doing the right sort of thing, because otherwise you could be doing them all wrong.

Attitudes to the graphic calculator were tested by questionnaire, as well as by interview, in the pilot study. Over 40% of the students rated the graphic calculator as "good" after their module, compared to nearly 30% rating it as "good" beforehand. About 20% thought them "boring" or "hard to use" both before and after the module.

During the student interviews in the Year 7 study, the nine pairs of students (three from each participating class) who audiotaped their discussions in the classroom were asked about their views on the graphic calculator. Most said it had helped them to find answers, and that they had found it useful. Only one pair mentioned any real problems. They said it was annoying to find the letters, and to remember the STORE and ALPHA keys. They also said it sometimes made things too complicated, and it was easier to do the questions in your head, comments which echoed those of Holly and Carly from the pilot study. It was observed throughout the teaching period that the majority of the students enjoyed working with the calculators.

In student interviews following the Year 8 study, both groups of students interviewed agreed that the graphic calculator was useful to help them check their answers. The higher achieving group¹⁸ said they could solve the equations without any extra help, but they liked to be able to confirm that their answers were correct as they worked through them. The

lower achieving group¹⁹ agreed with this, but said they would only check their answers once they had finished a set of questions. However, they felt that the calculator helped them directly through its feedback, rather than simply providing answers. This group did not have the certainty the first group had that they could solve the equations without help. The higher achieving group said the calculators were most helpful when they did something new, and that after they had done one or two of a given type of equation, they found it a bit annoying to have to use it to check their answers.

Students in the Year 6-8 survey were asked how helpful they had found the graphic calculator in a questionnaire at the end of the teaching period. Of the 272 who completed this questionnaire, only 10% thought it “not very helpful” or “not at all helpful”, with the vast majority finding it at least “OK”. Over 60% found it either “very helpful” or “quite helpful”.

4.5 SUMMARY AND CONCLUSIONS

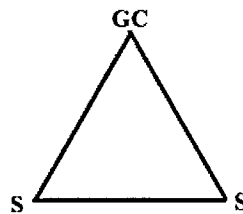
This chapter describes a model in which the graphic calculator provided a significant contribution to an environment in which students can begin their early studies of algebra. Such an environment is a social one, in which the calculator can be viewed as a mediating tool (Vygotsky, 1978; Wertsch, 1985). A student’s peers and the graphic calculator together form the scaffolding enabling a student to reach a higher level of understanding of

¹⁸ A group of four, chosen by their teacher as relatively “able” mathematically, and articulate.

¹⁹ A similar group of four, also articulate, but relatively “less able” mathematically, also selected by the teacher.

a variable than would have been the case if they had worked alone and without the graphic calculator:

Figure 12: Diagram illustrating ZPD formed by two students and a graphic calculator



The student-graphic calculator-student triangle is instrumental in allowing a ZPD to form in which the students can develop their understanding.

There are four aspects to how the calculator contributes to this environment. It acts as a tool, shaping the higher mental processes of the students (Vygotsky, 1978). Secondly, if students work together sharing a calculator, the calculator forms a focus for discussion which is also part of shaping their thinking (Graham, 1998). Thirdly, it provides students with an easily understood concrete model for how an algebraic variable functions, and which allows them to connect the abstract and the concrete (Noss and Hoyles, 1996). Finally, it provides immediate feedback, enabling the students to challenge their misconceptions and correct errors (Gage, 2001, 2002b).

In the next three chapters, the results of the case studies and the survey are discussed in depth, focusing on cognitive change in the students (Chapter 5), the progress the students made in working with expressions (Chapter 6), and the misconceptions the students showed (Chapter 7).

CHAPTER 5 EVIDENCE OF COGNITIVE CHANGE

In the previous chapter, it was suggested that using the graphic calculator could enable students to change how they interpret letters, that is, to make cognitive changes in their understanding of how letters are used. This chapter concerns such cognitive change, both what it is and how it may be recognised. In section 5.1, cognitive change and criteria for its recognition are discussed. The criterion chosen, which derives from Pea's work (1985; 1987) is that there should be evidence that the student's thinking has been restructured in some way. This is then exemplified with vignettes from the classroom case studies, which are described in section 5.2. The examples chosen include occasions where cognitive reorganisation did occur and also occasions where cognitive reorganisation did not occur for some reason. The chapter is summarised in section 5.3.

5.1 WHAT IS COGNITIVE CHANGE, AND HOW CAN IT BE RECOGNISED?

5.1.1 Metaphors for the effects of cognitive technologies

'Amplification' and 'cognitive reorganisation' are metaphors frequently used to describe how students might learn more or learn more quickly when using computer technologies (sections 2.5.2, 4.2.1). In 1985, Pea wrote that "[c]omputers are classically viewed as amplifiers of cognition", taking up what he called Bruner's "influential phrase" of "'cultural amplifiers' of the intellect" (p168). Computers could be expected "inevitably and profoundly [to] amplify human mental powers". This expectation of inevitability has since been challenged, particularly by the French school whose work was reviewed in

section 2.5.3, but the metaphor of ‘amplification’ to explain how computers might help us persists.

Pea used the idea of amplification as a metaphor for those interactions where the computer speeds up what the learner does, or extends what they can do. Such effects can be observed immediately, and occur *while* the learner is using the technology. An example might be that of the student who is assisted in grasping certain features of graphs by having a graphic calculator draw a series of graphs. Here the graphic calculator would enable the student to observe features across a series of graphs by removing the necessity to draw out all the graphs by hand.

However, Pea felt that the metaphor of amplification was inadequate to theorise the effects of cognitive technologies. In many cases, the effect of using computer technology is not simply to speed up a task, or to extend a person’s capability in some way, but to restructure the task. Pea gave the example of a young child who uses a pencil to help her/him remember a long list of items, suggesting that it would be:

... distortive ... to say that the mental process of remembering that led to the outcome was amplified by the pencil. ... The pencil did not amplify a fixed mental capacity called memory; it restructured the functional system for remembering, and thereby led to a more powerful outcome. (Pea, 1985: 170).

Instead, Pea proposed that the metaphor of cognitive reorganisation would be a better one to describe how “computer-based cognitive technologies such as software fundamentally restructure the functional system for thinking” (p170).

The effects of cognitive technologies are much more complex than the metaphor of amplification would suggest, because the nature of the learner’s thinking is changed (Pea, 1987; cf. Vygotsky, 1978: 132f). The specific details of how thinking is restructured are

unpredictable, with the emergence of new qualities in the learner's thought. Pea defined cognitive technologies as "any medium that helps transcend the limitations of the mind, such as memory, in activities of thinking, learning, and problem solving" (1985: 168). An example is the invention of written language, which enables us to rise beyond the restrictions of our memories, to externalise fleeting thoughts for subsequent reflection, analysis and discussion.

The crucial difference between 'amplification' and 'cognitive reorganisation' as metaphors to explain how cognitive technologies enable us to do more than we could previously, is that our partnership with cognitive technologies gives us the opportunity *to engage in new tasks, not simply to do old tasks better*.

... it might be said that the real power of technology ... is in its ability to redefine and fundamentally restructure *what* we do ..., *how* we do it, and *when* we do it. *We come to use this technology as a tool to think with.* (Salomon, 1991: 191, original italics)

Salomon identified the unit of analysis as not simply the individual, but the partnership between the individual and the intellectual tool. He argued that there is a need to distinguish between mental operations reorganised *during* this partnership, and those altered *as a result* of it (p192). Pea expressed this in the question: "How can technologies for education serve not only as tools for thinking, but for helping thinking to develop?" (1985: 178).

Following this line of thought, in this chapter I want to explore how the use of the graphic calculator can enable students to develop how they think about variables. I argue that such development in students' thought is caused *as a result of* the triad of two students and a graphic calculator, and is not simply something that occurs *while* they use the calculator. The calculator is a necessary part of the learning situation: its replacement by some other tool or form of technology would change the learning that occurred.

5.1.2 Cognitive reorganisation: Vygotsky

Pea's metaphor of cognitive reorganisation has its origins in the work of Vygotsky (Pea, 1985, 1987). Marx's theory of society (historical materialism) identified "historical changes in society and material life" as the causes of changes in human nature (Cole and Scribner, *Introduction to Vygotsky*, 1978: 7).

Our productive activities change the world, thereby changing the ways in which the world can change us ... (Pea, 1987: 93, original italics)

There is a dialectical process between us changing our world and our world changing us. According to Marx and Engels, labour is the factor which mediates our relationship to nature, and this relationship is fundamentally changed by the use of physical tools. Such tools enable us to change our environment, and, in turn, the environment changes us:

Vygotsky argued that the effect of tool use upon humans is fundamental not only because it has helped them relate more effectively to their external environment but also because tool use has had important effects upon internal and functional relationships within the human brain. (*Afterword*, Vygotsky, 1978: 132f)

Vygotsky and Luria generalised this theory and applied it to a historical analysis of symbolic tools, such as speech and writing. Vygotsky (1978) concluded from this that mental processes are mediated by signs (or symbolic tools) in the same way that physical processes are mediated by tools (even if only by a hand).

Because tools and signs have an inevitable effect on us through this dialectical relationship, it follows that amplification is not a sufficient metaphor to describe their effect, since amplification presumes that there is no change in how we think.

The growing complexity of children's behavior is reflected in the changed means they use to fulfil new tasks and the corresponding reconstruction of their psychological processes. (Vygotsky, 1978: 73)

From this, Pea (1985; 1987) concluded that cognitive technologies are reorganisers rather than amplifiers of the mind. The examples Pea (1987) listed (which echo those of

Vygotsky, 1978: 7, 38) include all symbol systems such as written language, mathematical notation, computer language, and physical technologies such as chalk and board (which has to be erased), pencil and paper (which does not), and so on. These all have the power to externalise the intermediate products of thinking, and to make them permanent for a greater or lesser time.

Suggestions that this can be interpreted as an amplification of the ZPD (Berger, 1998) seem to interpret the ZPD metaphor in too concrete a way. The amplification metaphor does not encompass new thought constructs, so the ZPD cannot be extended if there is only an amplification effect. The ZPD is the difference between what the child can do alone and what s/he can do with help, and it involves new ways of thinking. It is cognitive reorganisation that affects the ZPD, leading to a more powerful outcome than would have been the case without the cognitive technology which mediated the reorganisation.

5.1.3 The graphic calculator: a tool for cognitive reorganisation

The graphic calculator has been described as an intelligent technology “capable of significant cognitive processing on behalf of the user” (Berger, 1998: 14). However, the calculator can be used in a variety of ways, and it may be the case that some of these are more capable of such cognitive processing than others. Much research on their effectiveness in aiding learners relates to the use of graphic calculators in providing graphs, and is therefore not particularly relevant to its use in helping children learn about variables. It is argued here that the graphic calculator model of a variable is a form of cognitive technology in Pea’s sense: it is a tool which helps transcend the limitations of the mind. Its

affordances¹ enable children in the 10 to 14 year old age group to understand what letters used in algebra mean and how they are used.

A study designed to discover if the graphic calculator has detectable amplification or cognitive reorganisation effects was carried out by Berger (1998) with South African students, who were learning basic calculus. Her project was limited by the availability of the calculators, so only twenty students were included. These students had an additional 45 minute tutorial a week in which they were given guidance about using the calculator, but otherwise they followed the same course as the other students in their year. Since use of the calculators was not permitted in examinations at the time, use of the calculator was limited to verification and support of analytic results in these tutorials. Berger found amplification effects in her qualitative data, in that students were able to generate graphs quickly and easily for further consideration. She had hoped also to find evidence of cognitive reorganisation effects, but she was only able to report one such incident.

Berger's criterion for cognitive reorganisation was that the student would

... use her calculator in a manner which was qualitatively different from that possible with pencil and paper ... (p18).

It is difficult to see how this could be a useful criterion, however, since using a different tool means that there will almost certainly be differences in how the students carry out tasks. Berger's criterion is also very difficult to use in practice: how can one know if a given student's thinking is different using one cognitive technology from what it would have been had that student used a different technology? Pea's (1985) definition is that

¹ 'Affordances are the properties of a system ... which allow certain actions to be performed and which encourage specific types of behaviour' (Cox, Webb, *et al.*, 2004).

there is evidence of restructuring the functional system for thinking. Berger's and Pea's definitions are not necessarily the same: on the one hand, Berger was looking for evidence that the calculator has been used in a way that would show a different approach, and on the other, Pea was looking for evidence of restructuring of thinking. The criterion chosen here for cognitive reorganisation is that there is evidence of the student's thinking being restructured in some way, as this is more fundamental and easier to detect.

5.2 EXAMPLES OF COGNITIVE REORGANISATION

In this section, several episodes are described from the three case studies carried out in this research. The incidents are grouped together thematically, and include examples of occasions where there was an opportunity for cognitive reorganisation which did not take place. It is argued that the graphic calculator can provide a locus in which students are enabled to change their constructs, but only if the change required is within the students' ZPD. If it is outside the ZPD made up of the unit of the two students and the graphic calculator, then the anticipated cognitive reorganisation will not take place.

5.2.1 What does $2a$ mean?

This excerpt was recorded by two students during the first lesson in the Year 7 study, and shows the way in which the graphic calculator has the potential to enable students to change the way they think. The previous question asked them to evaluate $2 \times A$, given that A was 4, which they did correctly. They continued:

- Charlotte: Three is 2A. 2 and 4 must be 24, I think. *This is a perfect example of the 'code' misconception².*
- Abigail: ...
2B.
- Charlotte: ...
It'll be 24.
- Abigail: Yeah. No, this is B. *Referring to question number five.*
Five.
- Charlotte: 27, it's 27. *The value they were given for B was 7. Again, note the 'code' misconception.*
- Abigail: ...
Then eight, which is AB *Question 8.*
- Charlotte: ...
... and that will be 12 ... *It is interesting that Charlotte's first thought was to substitute A=1 and B=2, and to interpret each as a digit of a single number (the 'code' misconception again). However they settled for 47, again 'code', but this time using the given values.*
47.
- Abigail: 47. Then nine ... BA. *Question 9.*
- Charlotte: Yeah, so that will be 74. *They were still using the 'code' misconception.*
- Abigail: 74. ...
- Charlotte: ... We now have to check them all with the calculator.
- Abigail: ...
2 times ALPHA A, 8. *$2 \times A = 8$, since $A = 4$.*
We got that right, so we can write in 8.
- Charlotte: OK, now I'll do it ...
number three. So we got that one right, so ...
- Abigail: Three ... 2A ... so 2
ALPHA A ...
- Charlotte: So ... 2 ALPHA A *$2A = 8$ also.*
equals 8 ...
- Abigail: Whoops! *They realised that they have misinterpreted 2A, since they put 24.*
- Charlotte: [laughs]
- Abigail: OK, maybe I did it wrong!
- Charlotte: [laughs again, then a long pause] *They were clearly both stunned by this turn of events.*
- Abigail: OK, so that's 8. [tape turned off]

² 'Code' is a category of error (see Chapter 7: *Misconceptions*) where students interpreted a letter as one digit in a number rather than as the whole number.

It is a pity that this was the end of the lesson, and Abigail was away for the next lesson, so there is no record of how they sorted this out. However, compare the following extract from their discussions, taken from the transcript of the third lesson. This extract gives an example of how a gain in skills can be related to proceptual thinking (Tall and Thomas, 1991), and demonstrates the role of the graphic calculator in enabling this cognitive reorganisation to occur. The students were trying to find expressions equivalent to $2(S + T)$ from a list of possible choices, but at this stage they had done no work on multiplying out brackets. The correct choices given to them were $2S + 2T$ and $S + S + T + T$:

- | | | |
|------------|--|---|
| Abigail: | S plus T has to be done first. | <i>Because it is in brackets presumably.</i> |
| Charlotte: | That means 2 times S plus T .
S plus T times 2. | <i>Note Charlotte's recognition of $S + T$ as a procept.</i> |
| Abigail: | I can't find that. | |
| Charlotte: | Oh, there it is, S plus S plus T plus T . | |
| Abigail: | Is that right? Are you sure? | |
| Charlotte: | Yes, because it's 2 times S and ... I think so anyway. | |

They then checked that $2(S + T)$ equalled $S + S + T + T$ by putting numbers into the S and T stores on the calculator and evaluating the two expressions. They found them to be equal, so moved on, satisfied, to the next question. Although Charlotte's skills in manipulating algebraic expressions had yet to be developed in any real sense (she missed $2S + 2T$), she was already gaining a sense of S plus T as an encapsulated object that can be operated on. Both students showed how far they had moved from thinking that the A in $2A$ meant the second digit of a two-digit number. The graphic calculator model had enabled them to restructure their thinking.

5.2.2 Interpreting $2x$ and x^2

In this example, three incidents are described, two from the Year 7 study, and one from the Year 9 pilot study. The Year 7 students were struggling to interpret expressions like $2x$.

Does it mean $x + x$, or $x \times x$? The root of their confusion is that $2x$ means that x should be *multiplied* by 2. The Year 9 students were trying to interpret terms like $5x^2$, initially squaring the 5, rather than the x , to give $25x$.

Does $2P = P + P$ or $P \times P$?

Evidence that when students work collaboratively with a graphic calculator, cognitive reorganisation can occur can be seen in the problem of sorting out whether $2P$ is the same as $P \times P$ or $P + P$. This particular question came towards the end of the Year 7 teaching module, and caused confusion for many of the students who tried it. Students were asked to find equivalent expressions for $2P - Q$ from alternatives which included $P + P - Q$ and $P \times P - Q$.

This first excerpt is from Sofia and Chantelle's discussions. These were both students who had done some algebra in their primary schools, and who had made good progress during the Year 7 module.

- | | | |
|------------|---|--|
| Sofia: | Which of the following is the same as $2P$ minus Q ? | |
| Chantelle: | So ... | |
| Sofia: | One second. The choices are ... | |
| Chantelle: | The choices are P add P take away Q , or P take away Q plus P ... | $P + P - Q, P - Q + P \dots$ |
| Sofia: | P times P take away Q , P plus P plus Q , Q plus P minus P or ... | $\dots P \times P - Q, P + P + Q, Q + P - P \dots$ |

Chantelle: ... Q take P take P . $Q - P - P$
 ...
 Sofia: $2P$ equals P times P . *Sofia seemed very sure of her ground.*
 Chantelle: Yes. No. $2P$ equals P plus P . *Chantelle was equally sure, however.*
 Sofia: No. 2 times P , P times P . *This is the crux of the problem: $2 \times P$ is multiplication, and it is hard for a student at this early stage to see it as the sum of P and P , rather than their product.*
 Chantelle: Oh, right. Sorry.
 Sofia: But no. $2P$. *Maybe Sofia was not quite so certain.*
 Chantelle: So P plus P .
 Sofia: P times.
 Chantelle: Isn't it just $2P$, P plus P ? *Chantelle kept trying to bring them back to $P + P$.*
 Sofia: Not add.
 Chantelle: Yeah, but ...

Compare that with the following exchange between Megan and Lucy, who had also studied algebra before, and were getting on well during this module. These two students were in a completely different class, with a different teacher.

Megan: $2P$ is the same as P plus P . *Megan was in no doubt about this.*
 Lucy: No, it's not, it's the same as P times P . *Lucy was equally sure.*
 Megan: No, but ...
 Lucy: It's the same but ...
 Megan: Yes, I know, but plus ... and ... so it means $2P$.
 Lucy: Yes ...
 Megan: ... so they have to be added together, not taken away. *Megan knew what she was trying to say, but had difficulty articulating it.*
 Lucy: No, you have to times. *Again, this suggests the root of the problem is that $2P$ involves the operation of multiplication, not addition.*

Although these two pairs were from different classes, their exchanges were almost identical. Both conversations continued in a very similar way, with Chantelle and Megan trying to convince their partners that $2 \times P$ is $P + P$ not $P \times P$, and Sofia and Lucy appearing convinced, then saying "Yes, but ..." Both Sofia and Lucy appeared unable to

let go the idea that as $2 \times P$ involves a multiplication, so too should any equivalent expression.

This confusion is apparent in the discussion that ensued between Megan and Lucy:

- Lucy: Times or add then take Q .
 Megan: But there's two P 's but because to work it out you actually times it. *Megan was still trying to explain the confusion ...*
 You have like ...
 Lucy: 2 times P take Q is what you're saying, right?
 Megan: Yes. It is. P plus P take Q .
 Lucy: Yes. P times P . *... but Lucy remained quite sure that she was right (like Sofia).*
- Megan: Plus.
 Lucy: Times, it's times, but put ... oh right, fine. *However (like Chantelle), she appeared about to concede the argument.*
- Megan: Wait, can I just prove my point? *But Megan wanted to prove the point, not just win the argument.*
- Lucy: I know.
 Megan: Pretend P is 2. *Unfortunately, she started with the one value that would not help her ...*
- Lucy: You've got 2 plus 2 is 4 but 2 times 2 is 4 as well. I understand what you mean. *... as Lucy quickly pointed out.*
- Megan: But because ... Let's pretend P is 3. *So she moved on to 3.*
 You'd have to have two P 's.
 Lucy: Yeah, no, hang on, 3 plus 3 is the same as 3 times 3. No it's not. If it's 3 then 3 plus 3 is 6 and 3 times 3 is 9 ... so that's only working with ... no, it's not ... sorry ... you've got 3 plus 2, 3 times 2 ... but put in brackets ... *Lucy's confusion was growing. Having started correctly, by considering $3 + 3$ and 3×3 , she then started adding and multiplying by 2, and worrying about brackets.*
 um ... P times P take ...
 Megan: But ... um ... for example, pretend that P is 7. *Megan abandoned this and tried yet another value, this time 7.*
- Lucy: Yeah.
 Megan: You would have ... you would have two 7's.
 Lucy: Mmh hmm ...
 Megan: So it could be P and P or two times 7 ... or like 7 and 7.
 Lucy: Yeah.
 Megan: So that's the same as P plus P take Q .
 Lucy: I see what you mean ... cos P times *Lucy appeared about to*

- ... no ... *make the breakthrough ...*
- Megan: But if P is 2 ...
- Lucy: I understand because 7 times 7 ... no, 7 times 2 ...
- Megan: 2 is a number which ... 2 add 2 and 2 times 2 are exactly the same. *Meanwhile, Megan was explaining why 2 is a poor choice for this explanation.*
- Lucy: ... can be added or timesed ...
- Megan: ... so when you get to other numbers, like my example, 7, it would be 49.
- Lucy: No it won't, if you've got 2 times ... but P times P ... I see your point, yeah, you're right. Sorry. P plus P take Q . *Finally, Lucy did appear to have understood what Megan was saying.*

It appeared at this point that Lucy and Megan had resolved their difference by using

numbers to convince Lucy that her interpretation of $2P$ was incorrect. Yet, the final

resolution of this problem shows that Lucy was still not completely certain at this point.

Using the graphic calculator, they put 7 into the P store and 8 into the Q store, and showed that $2P - Q$ is equal to $P + P - Q$. Then:

- Lucy: Can we just do [the] one I thought it could be. *Lucy still wanted to check out $P \times P - Q$.*
- Megan: Yes.
- Lucy: ... which is P times P ... ALPHA P , times ALPHA P , take Q , ENTER. *Lucy entered $P \times P - Q$ into the graphic calculator, and found it gave a different answer from $2P - Q$.*
- Right, you are correct. I understand your point. Right, I've been doing this wrong.

On the audiotape, Lucy sounded much more convinced at this point than she had at the end of the discussion using numbers. The previous discussion had clearly gone a long way to persuade her, but this final demonstration really made the difference. The graphic calculator's immediate feedback enabled her to move from being almost sure to being very sure, so changing the way she viewed expressions like $2P$. This is an example of a student restructuring her thinking: at the beginning of this episode, she was convinced that $2P$ meant $P \times P$; at the end she was certain that it meant $P + P$.

Sofia and Chantelle's resolution of the problem was similar, although they did not go through a stage of trying out different numbers, but went straight to using the graphic calculator to evaluate the different expressions:

Sofia:	2 ALPHA P ...	<i>Sofia started the key sequence needed to find a value for $2P - Q$, having first put 1 into the P store and 2 into the Q store.</i>
Chantelle:	You're probably right though!	<i>Chantelle was prepared to concede the argument, although it is clear she was not convinced by Sofia's case.</i>
Sofia:	... take away ALPHA Q equals zero.	<i>Sofia obtained a value of 0.</i>
Chantelle:	Yay!	<i>Presumably Chantelle saw this as a vindication of her argument ...</i>
Sofia:	That's right.	
Chantelle:	Yeah, so it can't be zero.	<i>... although what she meant here is not clear.</i>
Sofia:	No, wait ... ALPHA P times ALPHA P take away ALPHA Q equals minus 1. Hmmm... Well! Which one do we go for now?	<i>Sofia then evaluated $P \times P - Q$, getting a value of -1 which is clearly not the same.</i>
Chantelle:	Exactly. Let me ...	
Sofia:	Let's try P plus P minus Q .	<i>She then tried $P + P - Q$.</i>
Sofia:	... ALPHA P minus ALPHA P ... sorry, it's plus ... ALPHA P plus ALPHA P take away ALPHA Q ... equals zero. So it must be P plus P . You're right! ... Pen eraser please!	<i>Sofia then evaluated $P + P - Q$, and obtained 0, the same as for $2P - Q$, and conceded the point.</i>

It is just as well that Sofia did not put 2 into the P store, or the contradiction would not have been exposed. Although it had been pointed out to the students that using 0, 1 and 2 could cause false results, these two had clearly forgotten this.

However, the final comments of both Lucy and Sofia show a remarkable similarity.

"Right, you are correct. I understand your point. Right, I've been doing this wrong." and

"You're right!" In the end, both are absolutely convinced, and have changed their

constructs for $2P$. In both cases, the graphic calculator provided the support they needed to abandon their old ideas and to accept what their partners were saying, with the calculator acting as a tool which helped the students to learn. The graphic calculator allowed the students to retain control of their own learning. They worked independently and co-operatively without any recourse to their teachers, with all four showing that they could negotiate their ideas and understanding quite successfully for themselves.

What does $5x^2$ mean?

In the Year 9 pilot study, some of the more capable students worked on simplifying expressions like:

$$3x + 5x^2 - 7x^2 - 6x \qquad 4x^2 - 7 + 4 + 5x^2$$

They had already simplified expressions with similar structures like:

$$2a + 3b - a - 4b \qquad 6c - 3c - 2d + 4d \qquad 6x + 3 - 4x - 7$$

In the episode described here, however, it is clear that one of these students at least did not recognise that all these expressions were essentially the same structure.

Lauren: For number four do you, like work out the square first, so do you put 25, so ... $5x^2$... isn't that 25x...

Gemma: What?

Lauren: ... and then...because 5^2 is 5 times 5, so isn't that 25x and then that's 49x, then you ...have 3x plus 25x minus 49x minus 6x...

Gemma: Yeah, or can't you just do $5x^2$ take $7x^2$, and do the 3x take the 6x separately?

Number four required them to simplify $3x + 5x^2 - 7x^2 - 6x$.

Lauren wanted to simplify $5x^2$ to 25x, and $7x^2$ to 49x ...

Gemma on the other hand suggested the correct method for simplifying this expression.

- Lauren: But it's all x 's, they don't make separate... *Lauren failed to recognise that terms in x and in x^2 should be treated separately, like the a 's and b 's in the previous examples. Her knowledge of how to deal with expressions like this is not sufficiently well formed to resist the destabilisation caused by the presence of the x^2 terms.*
- Gemma: But then if you add them together ... We'll just do it your way, because it's easier. *Unfortunately, Gemma was not sufficiently sure of her ground to overrule Lauren.*
- Lauren: OK.
- Gemma: OK.
- Lauren: So we are doing $5x^2$ equals $25x$...
- Gemma: Yeah.
- Lauren: ... and $7x^2$ equals $49x$.
- Gemma: Yes.

The discussion about this expression concentrated on simplifying the terms in x^2 initially.

Lauren reduced $3x + 5x^2 - 7x^2 - 6x$ to $3x + 25x - 49x - 6x$, which no doubt looked a lot more familiar to them, and was something they knew they could simplify. Although Gemma initially wanted to proceed by working with the terms in x^2 separately from those in x , she allowed Lauren to go ahead with her method, saying: "We'll just do it your way, because it's easier." They then simplified their expression to $-15x$, by adding the $3x$ and $25x$ to get $28x$, then subtracting the $6x$ from the $49x$ to get $43x$, and finally subtracting this from the $28x$ to get $-15x$, so they had actually simplified $3x + 25x - (49x - 6x)$. Having reached something that looked as if could be a final answer, $-15x$, they decided to check it with the graphic calculator³.

- Gemma: Minus 15X.
- Lauren: We're just going to check it on the calculator. OK. To check number 4 we are typing in ... *They typed in $3X + 5X^2 - 7X^2 - 6X$...*
- ...

³ In this study students were told to make sure they had values other than 0, 1 and 2 in the stores they were using.

Lauren:	... minus 6X, 2nd TEST, ENTER. What did we think our answer was?	... then the Boolean '=' sign (produced by 2 nd TEST, ENTER ...
Gemma:	Um, minus 15X.	... followed by their answer.
Lauren:	Minus 15X, ENTER. We got a zero, which means we got it wrong, we're just going to rework it out.	The calculator returned a 0, indicating that the two expressions were not equivalent.
Gemma:	OK.	

The calculator's immediate feedback showed them that there was a problem somewhere. It gave them no indication where the problem was (in fact they had made two different types of error), which allowed them to set about correcting their answer in their own way. They decided to go back to Gemma's suggestion, of working with the terms in x and x^2 separately:

Lauren:	OK. We're now going to try it by doing ...	
Gemma:	Um ... Yes, we're going to do 5X [sic] take 7X ² equals minus 2X ² ...	They reverted to Gemma's way of doing it, subtracting 7X ² from 5X ² to get -2X ² ...
Lauren:	... and 3X take 6X equals minus 3X...	... and 6X from 3X to get -3X.
Gemma:	... so minus 2X [sic] take 3X equals ...	
Lauren:	We didn't have time to work out number 4...	
Gemma:	... because we have to go home.	As they ran out of time, we cannot see if Gemma's mistake with 2X instead of 2X ² was a genuine mistake, or a slip of the tongue.

It is a pity that the two girls were not able to finish this. However, provided Gemma's 2x was a slip of the tongue, trying their new answer on the graphic calculator would have shown them that this time they were correct. Without the prompt feedback of the graphic calculator, they would probably have been quite satisfied with -15x, as it was a very simple answer. However, they found out immediately that they were wrong, so went back to the alternative suggestion, which they had initially ignored. Even though they were not able to verify that their second attempt was successful, the graphic calculator had enabled them to

reject Lauren's suggestion for dealing with the terms in x^2 . Gemma was not sufficiently sure of her method to over-rule Lauren when they started this the first time, and it had not occurred to Lauren to do it that way. This incident shows that both girls were well on the way to changing their constructs for dealing with expressions containing terms in both x and x^2 , and is thus very likely to have been another example of cognitive reorganisation.

5.2.3 Solving equations

In this section, students' efforts to solve three equations are discussed. The first equation proved to be a problem, probably because the way it was displayed on the graphic calculator screen meant that the students did not interpret it correctly. The second equation involved a negative coefficient of x . The third equation showed up an inadequate conceptualisation of ax , which meant that the students' method could not be extended to a case where a was not a whole number. In two of the three cases the graphic calculator was central in allowing the students to move on in their thinking; the third is an example of a case where the question is at present outside the ZPD created by the student-calculator-student triangle, and so cognitive reorganisation could not occur.

How do you solve $6x/4 + 9 = 4.5$?

This equation was also discussed in 4.2.2. In Chapter 4, the two Year 9 students from the pilot study who were discussed (Eleanor and Kerry) attempted to solve the equation by guessing, with the graphic calculator's immediate feedback eventually forcing them back on the method they had been taught. In the example analysed here, the two girls involved (Emma and Felicity) tried to use an appropriate method from the start, but made a mistake in its execution. The screensnap these two students were copying is shown in Figure 13.

Figure 13: Screensnap illustrating equation

The screenshot shows a calculator interface with a rectangular display area. Inside the display, the text "6X/4+9" is on the left and "4.5" is on the right, indicating a calculation result.

- Emma: OK, number 3. OK, this is $6X$ divided by 4 plus 9 equals 4.5. Well, 9 plus 4 equals 13. What's 4.5 times 13? 52, 52 ... 58.5.
- Emma started by misunderstanding the problem, working on $6x/(4+9) = 4.5$, rather than $6x/4 + 9 = 4.5$. Her misinterpretation was probably caused by the calculator's way of showing a division sign, so that the 4 was on the same level as the 9.*
- Felicity: Yeah. Oh, 5.
- Emma: 58.5 ...
- Felicity: You write in the number you think it is, then you press the STO button, then you press X,T ... it says on the sheet ...
- Felicity was looking at the instructions on the worksheet for using the graphic calculator to check an answer.*
- Emma: ...it's 9 point something, so ...
- This comes from $4.5 \times 13 \div 6 (= 9.75)$.*
- Felicity: Emma, explain! What are you doing?
- Emma: 9.5 right, STO X, ENTER, CLEAR, OK.
- Emma put 9.5 into the X store of the calculator.*
- Felicity: $6X$ divided by 4...
- Emma: $6X$...
- Felicity: ...divided by 4 ... divided by 4 plus 9, 2nd MATH, ENTER...
- 2nd MATH (or TEST), ENTER gives the Boolean '=' sign.*
- Emma: I think I'm totally wrong actually. ... plus 9, ENTER.
- Felicity: Yeah, you're wrong. So what were you using, 4.5?
- The calculator returned a 0, telling them that they were wrong.*

There was nothing wrong with Emma's skill in solving equations, but her first instinct was to add the 4 and the 9, so that she was in fact solving the wrong equation. She knew about the order of operations in an expression like $6x/4 + 9$, but it seems likely that the way in which the equation was displayed on the screensnap destabilised her knowledge. She was used to equations of this type looking like this:

$$\frac{6x}{4} + 9 = 4.5$$

where the temptation to add the 4 and the 9 is much less, rather than:

$$6X/4 + 9 = 4.5$$

The two girls went on to try various numbers, including 9.8, 9.3, and 9.75. They both agreed that the value they were looking for was 9 point something, because six nine's are 54, and 4.5 multiplied by 13 is 58.5 (solving $6x/(4 + 9) = 4.5$). By the time they tried 9.75 however, they were basically using trial and error – it did not seem to occur to them to evaluate $58.5/6$ on the calculator. Each time they tried a value, the calculator returned a 0, showing them that their answer was still wrong. Then came the moment when Emma saw what they were doing wrong:

<p>Emma: Oh! It's divided by 4 and then plus 9 so that is going to be, it's going to be 9 less than 4.5, which is ... minus 4.5, so something 6, 6 times something divided by 4...</p> <p>Felicity: ...equals minus 4.5...</p>	<p><i>This is the breakthrough, when Emma changed how she understood the problem.</i></p> <p><i>This is now a correct conceptualisation.</i></p>
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They then correctly simplified the equation to $6x/4 = -4.5$, and proceeded to calculate a value for x , which they tried out on the graphic calculator:

<p>Emma: OK, um, STO X ...</p> <p>Felicity: What are you trying?</p> <p>Emma: Minus 3. ... Yes!</p> <p>Felicity: Yay!</p>	<p><i>Emma then checked to see that their answer was at last correct.</i></p>
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In this example, the graphic calculator feedback supported the students while they reconstructed the way they perceived the equation. Unlike the previous examples, where one of the student pair was seen constructing knowledge they did not previously have, this example showed the students stabilising knowledge they did have, but which was not yet totally secure.

You can't take $5x$ from each side of an equation because you don't know what it is!

In the Year 8 case study, students worked on a range of equations, including $18 = 8 - 5x$.

The two girls in the following extract were quite capable of doing apparently similar

equations, having just solved $4 = 4x - 2$ and $7x + 1 = 2x - 3$. However, they perceived

$4 = 4x - 2$ as much easier, since the coefficient of x is positive, and even $7x + 1 = 2x - 3$

appeared more straightforward for the same reason.

- | | | |
|--|--|--|
| <p>Claire: You would have to add ... we have to add ... is it $5x$ we would have to add on the other side, so that 18 ...</p> | <p>Briony: What?</p> | <p><i>The equation they wanted to solve was $18 = 8 - 5x$. Initially, Claire considered adding $5x$...</i></p> |
| <p>Claire: Wouldn't we have to add $5x$ to that side and to that side, so that 18 plus $5x$ equals 8.</p> | <p>Briony: Then that would be 8 minus $10x$... if we added $5x$ to each side ...</p> | <p><i>... to both sides of the equation, to give $18 + 5x = 8$. She had almost done the question at this stage!</i></p> |
| <p>Claire: Oh, do we have to take it?</p> | <p>Briony: So if we take $5x$...</p> | <p><i>However, Briony, doing the same operation of adding $5x$, obtained $8 - 10x$ for the right hand side ...</i></p> |
| <p>Claire: No, because we don't know what $5x$ is, so ... we could take 8 from either side, so that would make it easier! 10 equals $5x$, so 10 divided by 5 is 2 ... so ...</p> | <p>Briony: So let's ... do you want me to try that? 2 STO, ALPHA X, ENTER ... 8 minus 5 ALPHA X, ENTER, equals minus 2 ...</p> | <p><i>... which confused Claire ... and then confused Briony also. Her thinking by then totally disorganised, Claire decided they could not work on a term for which they did not know a value. Instead, she tried to take the 8 from both sides, to give $10 = 5x$. The sign error here is not just chance, however. Students often forget that signs are attached to the term which follows them, and in removing the 8, Claire probably felt she had also removed the minus sign. Briony then put the value of 2 into the X store, and keyed in the right hand side of the equation. When she pressed the ENTER key, the calculator gave a value of -2 instead of the value of 18</i></p> |

they were expecting⁴.

Claire: So that can't be right ...

Claire almost did the question correctly in the first few seconds, but when Briony clearly did not understand what she was doing, Claire lost her way. Instead, she became so confused she decided that they could not do anything about the $5x$, as they did not have a value for it. Removing the 8 instead led them to a sign error, and eventually to a wrong answer, as confirmed by the graphic calculator. Like Emma in the previous example, Claire's knowledge was destabilised, this time by Briony's lack of understanding and her suggestions of alternative ways to proceed.

Briony then proposed that instead of trying to deal with the $5x$ or the 8, perhaps they should try for something smaller, like taking off two or three, as this might be easier. They continued by subtracting three from each side of the equation:

$$18 = 8 - 5x$$

giving:

$$15 = 5 - 5x$$

At this point, Claire suggested that x might be a negative number "because two minuses equals a plus, so it must mean that $5x$ equals minus 10". Here Claire is using common sense, noticing that the $-5x$ term has to equal 10, or, equivalently, that $5x$ should equal minus 10. Briony mistook what she meant, and thought she was suggesting 10 as a value for x . The conversation then continued:

Briony: I don't know, I don't understand
what ...

*Briony did not
understand what Claire*

⁴ This was a simpler way of checking equations than using the Boolean '=' sign, as it did not involve using any additional menus, but just keys to be found on the calculator keyboard. In addition, problems encountered if the students used values of 0, 1 or 2 in the stores did not occur.

- Claire: Minus 2, I think it's minus 2, minus 2 STO, ALPHA X, ENTER then we do 5 take away, 5 ... *had just said. Meanwhile, Claire was testing her value of -2 in their revised equation.*
- Briony: Yeah, you're right. So what did we have to do to get that? *The calculator confirmed that $x = -2$ is the right answer, but Briony still could not see where it came from.*
- Claire: Um ... divide minus 10, because it's got to be two minuses, and minus 10 add minus 5 equals minus 15, so minus 10 divided by ... um ... 5 equals minus 2, and x equals ... *Claire's explanation was not a masterpiece of clarity!*
- Briony: So what is it? Divide minus 10 by 5, which equals ... *Briony seemed to have understood at least that they needed to calculate $-10/5$...*
- Claire: ... minus 2 ...
- Briony: ... so x equals minus 2. *... hence getting an answer of -2.*

Having discovered that their first answer was wrong, Claire and Briony managed to find a way that made sense to them, although it was not the most efficient method to solve the equation. At the heart of this process was the triangle of the student-calculator-student, forming a ZPD in which they could reconstruct how they understood the question leading to cognitive reorganisation.

5.2.4 Brackets

Expressions like $3 \times (A - B)$ and $3(A - B)$ were used in the Year 7 case study: students were asked to evaluate these expressions, with values of $A = 4$ and $B = 7$ using the graphic calculator, so that they could discover that the presence or absence of the '×' sign makes no difference. Equations like $4(2x + 1) = 10$ featured in the questions the Year 8 case study students worked on during lessons, and also during interviews held afterwards. The examples given below are both from the transcripts of Claire and Briony's discussions, the first when they were in Year 7, and the second a year later when they were in Year 8.

Claire and Briony, Year 7

In this first excerpt, Claire and Briony showed that they realised that $3 \times (A - B) = 3(A - B)$, but still managed to go astray because of the way that they managed the presence of brackets in the expressions.

- | | |
|--|---|
| <p>Claire: Now we're doing 3 times A take B, so first we have to do ALPHA A, take ALPHA B ...</p> | <p><i>They were working on $3 \times (A - B)$, with $A = 4$ and $B = 7$. Claire decided she needed to put $A - B$ into the calculator first as it was in brackets.</i></p> |
| <p>Briony: ... and we predict that will be 6 ...</p> | <p><i>They had already predicted that $3(A - B)$ would be 6. To do this they had multiplied -3 (from $4 - 7$) by 3, getting an answer of 6 (a mistake made by others also – they all started with -3 then added on 3 lots of 3).</i></p> |
| <p>Claire: ... times ... 3, equals ... um ...</p> | <p><i>Claire then evaluated $A - B \times 3$ with the calculator ...</i></p> |
| <p>Briony: ... equals minus 17, which is a slight problem.</p> | <p><i>... getting -17 (from $4 - 7 \times 3$).</i></p> |
| <p>Claire: So I think we got that wrong once again. Maybe we should check the rest of our answers. We got the next one right, because we had minus 17, so maybe we just wrote down the wrong number.</p> | <p><i>The wrong answer they were referring to was their prediction of 6. The next question was $3(A - B)$, which they predicted to be -17. They concluded that this was correct since they knew it would be the same as $3 \times (A - B)$. They then decided that their wrong answer of 6 was perhaps something they had written down wrongly. Here incorrect use of the graphic calculator led to it supporting an incorrect conclusion. The girls did not allow their initial value of 6 to challenge what they saw on the calculator, and so did not rethink these questions.</i></p> |
| <p>Briony: Yeah.</p> | |

Claire and Briony, Year 8

Contrast Claire and Briony again, one year later (in the Year 8 case study), working on a structurally equivalent expression, but this time in the context of an equation,

$$4(2x + 1) = 10:$$

Claire:	We could take 4 ...	<i>Confusion between division and subtraction was quite common, and occurs elsewhere in the transcripts from the Year 7 and 8 case studies.</i>
Briony:	No, we've got to do the brackets first, so it will be $2x$ plus 1	<i>Briony, thinking procedurally, wanted to do something with the brackets first. The conversation continued along these lines for a short time.</i>
Claire:	... because you've got to have four lots of the brackets ...	<i>Then Claire looked at the equation again, and observed its structure: it is 4 lots of something defined by the brackets.</i>
Briony:	Wait a minute. OK, so what we could do is ... um ... we do 10 divided by 4 ... which is ... can you just work that out	<i>Briony then took a second look, and saw what she meant. Having realised that the overall structure is a simple multiplication, she then saw that the required operation is to divide the 10 by the 4, giving 2.5. Continuing from this point, they evaluated $x = (2.5 - 1)/2$, getting a value of 0.75.</i>
Claire:	So let's try that. 0.75 STO, ALPHA X, ENTER, and then it's 4, open brackets, 2 ALPHA X, plus 1, close brackets, ENTER, is 10, so that's right. So x equals 0.75.	<i>They used the graphic calculator appropriately to check this, by putting the value of 0.75 into the X store, and evaluating $4(2x + 1)$, which confirmed that this was the right value for x.</i>

It is clear that in the year between the Year 7 and Year 8 studies, these two students had begun to think proceptually, and were able to use this more flexible approach in how they interpreted algebraic expressions and equations. This time they used the graphic calculator appropriately, confirming that they were correct. These two students had made a cognitive reorganisation in the course of the year, supported by the graphic calculator model of a variable.

Year 8 case study interviews

Procedural and holistic approaches to equations of the type $a(bx + c) = d$ were also apparent in the student interviews at the end of the Year 8 case study. The group deemed higher achieving by the teacher could see the structure of the equation, and hence deal

appropriately with the brackets, whereas most of the group deemed lower achieving worked in a purely procedural manner. The first excerpt is from the interview with the higher achieving group⁵:

- | | |
|---|--|
| <p>Anna: How would you do this equation,
 $4(a + 1) = 20$? Um ... I think you would do ... well, 4 times something equals 20, so it would be 4 times 5. Then you need to work out something plus 1 equals 5, so would it be $a = 4$. Then you do 4 plus 1 equals 5, then 4 times 5 equals 20?</p> <p>Group: Yeah.</p> <p>Claudia: Or could you do it like a balancing scale, so you divide by 4 on both sides, so that's just a plus 1, and the other side's just 5. Then you take off 1, so then one's 4, and the other's a, so it's $a = 4$.</p> <p>Group: Yeah.</p> | <p><i>Anna clearly saw $(a + 1)$ as an entity in its own right, multiplied by 4.</i></p> <p><i>Claudia's alternative was to use the 'balancing' method.</i></p> |
|---|--|

The other students then discussed which of these two methods they would use, with most choosing the 'balancing' method they had just been working on in the classroom in this study. These are both holistic methods, however, with the whole equation considered in both cases. Contrast this excerpt from the interview with the lower achieving students:

- | | |
|---|--|
| <p>Amanda: How would you do this equation,
 $4(a + 1) = 20$? ... take away the 1 ...</p> <p>Bethany: ... if you do that, that means 4 times ...</p> <p>I⁶: I think you are saying different things, so let's get you separately. Amanda, can you say your bit first, and then Bethany, you can do yours?</p> <p>Amanda: Um ... what I do is ... take away 1 from 20, to make 19, because it's add there, so it would just leave you</p> | <p><i>Amanda wanted to start by reversing the addition. This may be because it is in brackets, or it may be just that it is the operation that</i></p> |
|---|--|

⁵ Questions were given to the students on cards, turned face down. Students took it in turns to take a card, read out the question, and start answering it. The others then joined in, or were asked for their views.

⁶ The interviewer, I, was me.

- with $4a$, and then you do 4 ... would you do 4 divided by 20? appears 'nearest' to the a .
- I: So you would take the 1 first ...
- Amanda: Yeah, to make it 19 ...
- I: ... and then you would divide by the 4?
- Amanda: Yeah.
- I: Right. Bethany, you weren't saying quite that, I don't think, what were ...
- Bethany: No. I would do ... well, if you take away the 1, like Amanda said, that would be 19, but then wouldn't that leave you with 4 times a ? Bethany repeated the difficulty she saw in the method just explained, although her concern was still not made completely clear. It could be that she was worried about ending up with $19/4$, which as it is not an integer was perhaps not right.
- I: Mm.
- Bethany: ... and so 4 times a would be ... so yeah, you'd do the same ... you'd divide it, yeah.
- I: What would you do first, Donna?
- Donna: I think I'd probably take 1 from 20 ... Donna agreed with Amanda ...
- I: So you'd start that way as well?
- Donna: ... and that would be 19, then 4 times but she also had a problem with where that took her.
- I: Mm-hm. How about you, Freya?
- Freya: Well, I'd just do 4 times something add 1 equals 20, so I would do ... um ... well, I'd sort of experiment what you added 1 to and timesed it by 4 to make 20. Freya, on the other hand, showed signs of viewing the equation holistically ...
- I: So if you experimented, what sort of number would you start with, do you think?
- Freya: Well, 4, because it's quite easy. ... and was able to see what the answer should be, as it is straightforward.

Where all the first group viewed the equation holistically, using a method which is proceptual in its conceptualisation, most of the second group worked on reversing one operation at a time, floundering when the result they obtained was not a straightforward

calculation. In this group, only Freya was beginning to take a more holistic approach, and even then, she still relied on experimentation to an extent.

The teacher was interviewed a little later the same day. Questions about how the students used the graphic calculator to check their work were used to lead into a discussion about whether they would have been able to use it to detect errors in how they worked with brackets:

- T⁷: A lot of them didn't stay with the graphic calculator, when they were checking their answers really. They put in the value just to check whether the answer was right ... um ... yeah, they didn't go into the method of storing, putting the value into STO ...
- The teacher confirmed that she had not given the students any particular directions about using the graphic calculator, apart from reminding them of how they had used it the previous year, in the Year 7 study.*
- ...
- I: But if they got that one ... $4(a + 1) = 20$...
- T: Yes.
- I: ... if what they had done first was subtract the 1 and then divide by the 4, if they then repeat that on the calculator, they will merely repeat their error...
- This is the method preferred by three of the lower achieving group.*
- T: Yes.
- I: ... so did you find that that happened?
- T: Um ... the most successful ones, who were the ones who said that the thing in the brackets must be worth 5 ...
- As Anna had described in the interview with the higher achieving group.*
- ...
- ... and there was still a lot of them who would subtract 1, then divide by 4.
- The teacher agreed however that a number of the students would not have been doing it in a way that would be successful ...*
- I: But would those people have discovered that they had made a mistake?
- T: Yes, if they put it in the calculator. ... however, she appeared not to

⁷ T is the teacher; the interviewer, I, was me again.

- have realised that the inadequacy of their interpretation of the question also meant that their check would be inadequate, unless they were using the calculator correctly.*
- I: ... if you subtract 1 and divide by 4, you're going to get $3\frac{3}{4}$, 3.75 ...
- T: Yes.
- I: ... if they then did 4 times 3.75 plus 1 ...
- T: ... but they won't, they'll put the brackets in, you see ... *She was convinced that they would put the brackets in ...*
- I: Right.
- T: ... when they substitute the answer in they will type in 4 bracket ... *... repeating this again.*

To the experienced user of algebra, there is no problem with questions like these: the brackets signify that $(A - B)$, $(2x + 1)$ or $(a + 1)$ are mathematical objects which are multiplied by some number (which of course also means that the addition/subtraction takes precedence over the multiplication). To the novice user of algebra, using the ideas they have brought with them from arithmetic, the brackets signify that the calculation in the bracket is to be done first, then the result multiplied by the given number, which sounds almost like the same thing. However, the less successful students in these examples did not understand the sequence of operations the calculator would follow if the brackets were not actually used. This is illustrated particularly well in the discussion Claire and Briony had in the Year 7 study, when they obtained a result of -17 (discussed at the beginning of this section). They clearly had no idea where this had come from, or why it was so different from their predicted answer of 6 (which admittedly was wrong also).

The graphic calculator model depends on the calculator being used in the appropriate way if it is to help students reconstruct their thinking, with numbers put into the appropriate store. If it is just used like a scientific calculator to perform calculations, it is of very little use.

5.2.5 When the divisor is greater than the dividend

Various equations of the form $a = bx$, where $b > a$, and $a/x = b$, were set for the students to solve in the Year 8 case study. In the first set of questions, the first of this type was $10/m = 4$, which Fran and Rebecca managed to solve:

- Fran: $10/m = 4$. 10 divided by something equals 4.
- Rebecca: So if we do 4 ... 10 minus 4, which is 6 ...
- Rebecca was confusing division and subtraction, perhaps because she was happier with subtraction as a means of reducing the size of a number, or perhaps just because 10 is not a multiple of 4, so subtraction allowed her to give an integer answer.*
- Fran: No, we don't need to do minus 4, do we? You can't ... divide ... 4 only goes into 10 two times, and then there's a remainder of 2, so it must be a point number.
- Fran struggled to divide 10 by 4 ...*
- Rebecca: 10 times ... no ... um ... 10 divided by 4 ...
- Fran: ... we can do that, 2.5 ...
- Rebecca: ... 2.5 ... and then ... um ...
- ... eventually getting there.*
- Fran: ... and then 10 divided by 2.5 is equal to ...
- Rebecca: m must be equal to 2.5.
- Fran: m equals 2.5.
- Eventually they both concluded that m must have value 2.5. They checked this with the graphic calculator, simply by repeating the calculation $10/4 = 2.5$, rather than by putting 2.5 into the M store, and evaluating the left hand side of the equation.*

On the next question, $3 = 12n$, however, they hit a stumbling block:

- Fran: $3 = 12n$, so 12 times n equals 3. So it must be 12 times minus something.
- Again, trying to reduce 12 by subtraction rather than by multiplication.*
- Fran: Um ... I reckon it's 3 equals 12 and then the n is ... minus 9, that's my guess. I think n is minus 9.
- Rebecca: OK, so we put n is minus 9.
- They were both quite happy with this answer, since in their*

*experience so far subtraction
reduces the size of a number.*

When they returned to this question later, the conversation continued:

<p>Rebecca: The next one is $3 = 12n$, so it's 3 equals 12 something, and it can't be 12 times something, because then it would be bigger than 12, instead of less, so we are guessing that the n is a minus number, so if we do ... well to get from 12 to 3 we have to take away 9, so we're guessing that the n is minus 9.</p>	<p><i>Rebecca reiterated the argument for n being minus 9, unable to conceive of a multiplication that would cause the answer to be smaller. They presumably checked the arithmetic with the calculator which did not show up the error. They clearly did not put the value of minus 9 into the N store on the graphic calculator, and then evaluate $12n$.</i></p>
---	--

The following day, they were given $7/p = 14$ to solve and their confusion became yet more obvious:

<p>Rebecca: 7 divided by something equals 14. Um ... 7 divided by ...</p>	
<p>Fran: For the 7 divided by p equals 14 question, the only thing we can think of, as 7 is a smaller number than 14, is that the number is divided by a plus number which is minus ... and so we think that 7 divided by p, the p is plus 7, but ... we don't ... as I don't know if you can actually get plus numbers ... Rachel, can you get plus numbers? OK, we have no idea about number 5, so we'll just move on, and we'll do number 4 and number 5 again at the end.</p>	<p><i>Fran showed the extent of their confusion ... referring to 'plus numbers which are minus' ... and then asking someone else if you can have 'plus numbers'.</i></p>

Right at the end of the lesson, they came back to this question. Fran started by saying that she thought it was impossible, but someone else in the class then told her that x was 0.5, to which she replied: "7 divided by 0.5 equals 14? Oh, yes, it does! Wow!" Rebecca then checked this with the graphic calculator, this time putting 0.5 in the P store, and evaluating $7/P$, to find that this was indeed correct. It seems unlikely that simply being given the answer like this would have helped them to reconstruct their thinking however.

5.3 SUMMARY AND CONCLUSIONS

This chapter started with a consideration of what cognitive reorganisation might be, and how it might be recognised. Cognitive change or reorganisation is considered here to be a restructuring of a student's thought processes, as argued by Pea (1985; 1987). Examples were then given of cognitive reorganisation occurring during the classroom studies as a direct result of the support given to the students by the graphic calculator while they struggled with the questions they were doing. Examples of cognitive reorganisation failing to occur were also given, where the students did not have an appropriate method for a question, and/or did not use the graphic calculator in the appropriate way. Some of these examples show instances where the question is outside the ZPD defined by the pair of students and the graphic calculator. This is particularly apparent in the excerpts from the conversations between Rebecca and Fran.

Transcripts from the classroom studies showed two pairs of students, Sofia and Chantelle, and Megan and Lucy, sorting out for themselves without teacher intervention that $2x$ is equal to $x + x$, not to $x \times x$, even though $2x$ involves multiplication. Further examples showed students realising that they had made errors in solving various types of equations and in coping with the presence of brackets in an expression or equation. Claire and Briony were shown in Year 7 failing to deal appropriately with brackets, then in Year 8, finding a better way of working with them.

In most cases, we saw the graphic calculator providing immediate feedback which allowed the students to restructure their thinking. In some examples, however, students did not use the calculator in an appropriate way, and so such restructuring did not occur. Rebecca and Fran's conversations showed this, as did the Year 7 discussion between Claire and Briony.

In conclusion, I would argue that using the graphic calculator model of a variable can help students to reorganise their thinking in some of the tasks commonly given to 11 to 14 year olds. Working with the graphic calculator allows them to reorganise their knowledge structure in this area, provided it is used in a way that takes advantage of the model of the lettered stores, and is not just used to repeat calculations which may have been done incorrectly.

CHAPTER 6 DEVELOPMENTS IN STUDENTS' UNDERSTANDING AND SKILLS

6.1 INTRODUCTION

The focus of this chapter is on the progress students made in developing their understanding and skills in working with algebraic letters and expressions during the time they worked on the graphic calculator modules. As discussed in section 2.6, to be successful, students of algebra need a robust concept of a variable. They need to understand the nature of a variable and the operations on it, and to be able to perform algebraic procedures fluently, if they are to progress beyond the basics of algebra. It is a contention of this thesis that using the graphic calculator to begin the study of algebra will help a student to gain such a robust concept.

According to Vygotsky, the process by which conceptual knowledge is gained is complex. As described in section 2.4.4, he suggested that students learn both 'spontaneous' concepts through their everyday experiences, and 'scientific' concepts through instruction in school. These need to be brought together and to interact if a stable concept is to be formed.

Though scientific and spontaneous concepts develop in reverse directions, the two processes are closely connected. The development of a spontaneous concept must have reached a certain level for the child to be able to absorb a related scientific concept. ... It [an everyday concept] creates a series of structures necessary for the evolution of a concept's more primitive, elementary aspects, which give it body and vitality. Scientific concepts, in turn, supply structures for the upward development of the child's spontaneous concepts toward consciousness and deliberate use. (Vygotsky, 1986: 194)

For example, the concept of 'brother' is one which a child understands well from ordinary family life, yet it can be difficult for a child to define clearly. On the other hand, a child

may be able to define perfectly a concept taught in school, such as Archimedes' law, yet have very little real understanding of what it means (Vygotsky, 1986: 158).

For a stable, useful concept to be formed, the everyday and the taught need to mesh, so that the child has both the words available to discuss the concept and the experience which will give the words meaning for her/him. The purpose of the graphic calculator model and method described in earlier chapters of this thesis is to give students experience of letters at the 'everyday' level (that is, as a 'store' for numbers) and for them to connect this with the material taught in school. If this connection is successful, then students will understand better the concepts taught in school, and will be able to work with letters in algebraic contexts in a more meaningful way.

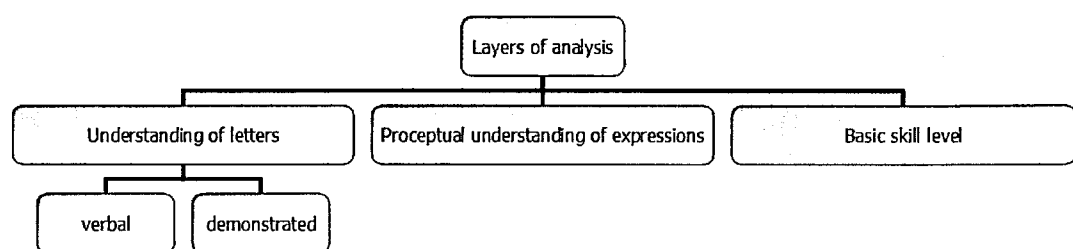
This process is theorised in this thesis using Vygotsky's theory of the mediation of tools and signs in the development of concepts. Vygotsky believed that humans form concepts through their interaction with both physical tools and symbolic or psychological signs, such as language. He did not view this as a one-way process, but saw such interaction as a dialectic process, with a person acting on a tool or sign, and that tool or sign simultaneously acting upon the person. The mediation of a tool or sign leads to change in the individual, as well as enabling the individual to change their environment.

In the context of the classroom work described in this thesis, the graphic calculator acts as both a mediating physical tool and a mediating psychological sign. Its function as a physical tool is more obvious: the students pressed keys on their calculators in order to copy the screensnaps they were given. In doing so, they began to internalise the idea that a number can be put into any of the calculator stores, and that this store is represented by an

arbitrary letter (sections 4.2.2, 4.4). Operating on these labelled stores gave the students experience in operating on variables, since these are isomorphic processes, and the calculator display is identical to written algebra. The graphic calculator's function as a psychological sign is perhaps less obvious. Examples Vygotsky gave of such 'signs' are the use of notched sticks or writing to help people remember things (van der Veer and Valsiner, 1994: 143). The graphic calculator display acts as a 'sign' in a similar way: it shows the student the result of an operation on a labelled store, and so helps the student to begin to understand algebraic operations.

To see whether development in understanding and skills occurred, three layers of analysis were conducted, as shown in Figure 14.

Figure 14: Analyses discussed in Chapter 6



The first layer concerned students' understanding of letters, which was tested verbally and by considering the understanding demonstrated in answering algebraic questions. Verbal testing was carried out using direct questions which asked what students thought the letters in an algebraic question meant. Students' responses were then compared with their demonstrated understanding. This was examined by comparing questions used in the questionnaires with those used by Küchemann (1981). His analysis was then used to decide on the level of understanding students demonstrated when actually working on

questions. For many students, their verbal responses were comparable to the level of questions they could tackle successfully, but there were groups where one was at odds with the other: this is discussed in section 6.3.3.

In section 6.4 the second layer, that is, the progress students made in their proceptual understanding of algebraic expressions, is considered. Here understanding is explored from a different perspective, using an analysis dependent on Tall and Thomas' (1991) definition of 'proceptual' understanding. This focuses on students' ability to perceive both the operations inside an expression and the expression itself as a holistic, mathematical object upon which further operations can be made. Such a flexible understanding of expressions is necessary for students to move beyond the most basic level in working with algebraic expressions (section 2.6.5). Some progress occurred in students' proceptual understanding, particularly among the students whose level of understanding was least.

The teaching modules used concentrated on giving students an understanding of what letters mean and how they are used, rather than on basic skills *per se*. It was hoped, however, that such an emphasis would also lead to progress in basic procedures, such as simplifying expressions, and this is discussed in section 6.5. It was found that most students did indeed make gains in the level of their basic skills, but those whose previous level of understanding and achievement was least made very good progress, whereas those displaying higher levels of understanding made less progress.

Finally, the chapter is summarised and concluded in section 6.6.

6.2 DATA USED FOR THE ANALYSES

In this chapter, data from the classroom studies carried out during this research project are analysed, in order to see if the graphic calculator did in fact enable the students to find greater meaning in the algebra they were studying. The data are also used to see if students developed greater skills in working with algebraic questions, as would be expected during more traditional algebra lessons. These analyses are used both to comment on Vygotsky's theory of conceptual development, and to assess the usefulness to teachers and students of the graphic calculator model and teaching method. The data used in these analyses were the questionnaires given to all the students before and after they did the classroom work.

The classroom studies conducted for this thesis were carried out in two phases, summarised in Table 5¹:

Table 5: Summary of phases of data collection in the classroom research

	Phase I			Phase II
	Pilot Case Study	Main Case Study		Survey
		Initial stage	Follow-up stage	
<i>School(s)</i>	← School G →			A, B, C, D
<i>Type of school(s)</i>	← Girls, selective →			Mixed, non-selective
<i>Date</i>	June/July 1999	Oct/Nov 2000	Oct/Nov 2001	Spring/Summer 2002
<i>No. classes</i>	1	3	1	12
<i>No. teachers</i>	1	3	1	6
<i>No. students</i>	30	79	28	307
<i>Age of students</i>	13-14 years, Y9	11-12 years, Y7	12-13 years, Y8	10-13 years, Y6-8

Phase I consisted of three case studies, with students from school years 7, 8 and 9. The students in the Year 8 follow-up study were also part of the Year 7 case study, conducted a year earlier. All these students came from one selective girls' school, school G.

¹ This information is discussed in more detail in Chapter 3: *Research methodology and methods*.

Further details of the students who participated in the Phase II survey are given in Table 6. These students were from four different non-selective schools, and from school years 6, 7 and 8. The figures shown without brackets are the proportions of students in each school and year group expressed as a percentage of the total number of students taking part in this survey (307). Figures in brackets are the actual numbers of students in each category.

Table 6: Details of schools and year groups involved in the Phase II Survey

<i>Year group</i>	<i>School A</i>	<i>School B</i>	<i>School C</i>	<i>School D</i>	<i>Sub-totals</i>
Y6	-	(36) 11.7%	-	-	(36) 11.7%
Y7	-	(109) 35.5%	-	(30) 9.8%	(139) 45.3%
Y8	(41) 13.4%	(69) 22.5%	(22) 7.2%	-	(132) 43.0%
Sub-totals	(41) 13.4%	(214) 69.7%	(22) 7.2%	(30) 9.8%	(307) 100%

In Phase I of the classroom research, questionnaires were administered to the students immediately before and immediately after their classroom work, but delayed post-questionnaires were not used. In the Phase II survey, pre-questionnaires and immediate post-questionnaires were administered immediately before and after the classroom work, with additional delayed post-questionnaires completed some four to six weeks after the end of the classroom work.

6.3 STUDENTS' UNDERSTANDING OF LETTERS

Determining students' understanding from their written answers to questions cannot provide indisputable results. Classifying students' answers is sometimes subjective, and so is deciding which questions should be considered. Although I have attempted to use the questionnaires that the students in the case studies and the survey completed to assess the level of their understanding of letters, both verbal and demonstrated, changes in the figures

produced for this assessment should only be considered significant where there is substantial change.

In section 6.3.1, students' responses to a direct question about their interpretation of letters in algebra are used to estimate their verbal understanding (corresponding to Vygotsky's scientific understanding). Then in section 6.3.2, an analytical framework provided by Küchemann (1981) is used to determine the level of algebraic question the students can actually tackle successfully. These two different facets of understanding are then brought together in section 6.3.3. In this section, the students are divided into subsets according to their verbal understanding of letters, so that a comparison of the levels of questions they can answer correctly can be made between the two subsets. I then use this comparison to comment on Vygotsky's remarks about the need for more than verbal or 'scientific' knowledge, if students are to form sound concepts. Finally, since this is a substantial section containing several different arguments, I have summarised the main points made in section 6.3.4.

6.3.1 Students' verbal understanding

Students' verbal understanding of what letters mean when used in algebraic expressions was tested using a variant of this question:

What do you think the a and c in question X mean?

The ability to answer this question satisfactorily corresponds to Vygotsky's hypothetical child who can state Archimedes' Law (Vygotsky, 1986: 158). On its own, possessing such verbal knowledge is insufficient for any real understanding of the underlying concepts.

This question was asked on all the case study and survey questionnaires, except those used in the Year 9 pilot study. Responses were divided into three categories: 'algebraic', 'numeric', and 'other'. 'Algebraic' covered any answer referring to numbers in general, or to 'different' or 'unknown' numbers. If specific numbers were mentioned, these needed to be purely illustrative. Examples of responses in this category were:

Letters in the place of numbers. *[Year 7 student, delayed post-questionnaire, school B]*

An unknown number. *[Year 7 student, pre-questionnaire, school G]*

A number. *[Year 8 student, pre-questionnaire, school B]*

I think they stand for different amounts of your choice. *[Year 8 student, delayed post-questionnaire, school C]*

Although such answers might indicate a good understanding of how letters are used in algebra, they could also conceal a multitude of misconceptions. A correct answer could mean that a student understood letters to represent numbers and was able to use this information in answering an algebraic question. Equally, it might indicate a learned response to the question, which did not influence the student's thinking when s/he worked on expressions. Assessing the extent to which this occurred is part of the analysis discussed in section 6.3.3.

The 'numeric' category comprised responses where specific numbers were used to define the letters, and not merely as an illustration, for example:

a means 1 and c means 5. *[Year 6 student, immediate post-questionnaire, school B]*

$a = 2$, $b = 3$. *[Year 7 student, pre-questionnaire, school D]*

$a = 1$, $b = 2$. *[Year 7 student, pre-questionnaire, school G]*

e.g.² $a = 10$, $\therefore (6 \times 10) + (2 \times 10) = 80$. e.g. $b = 1$, $\therefore (12 \times 1) - (2 \times 1) = 10$. [Year 7 student, pre-questionnaire, school G]

I think it means 0 and 5. [Year 8 student, pre-questionnaire, school A]

These students at least knew that the letters represented numbers, but had not yet realised that they can represent any number. $a = 1$, $b = 2$, $c = 3$, and so on, were particularly common responses in this category. Some students thought values should be substituted for letters, so that, for instance, $6a$ would be equated to 6 and $6b$ to 12; others used them as place-holders for digits, so that $6a$ would be equated to 61 and $6b$ to 62.

The 'other' category of answers consisted of responses that did not refer to numbers at all, including responses that were incomprehensible, and non-responses where students put 'don't know' or simply left the question blank. Examples of responses in this category are:

It means that it's an algebra question. [Year 6 student, immediate post-questionnaire, school B]

I think a means above a denominator. I think b means below a numerator. [Year 6 student, pre-questionnaire, school B]

I don't know but I know how to work it [sic] at most questions. [Year 7 student, immediate post-questionnaire, school B]

I think the a and b mean different angles, sizes or things. [Year 7 student, pre-questionnaire, school D]

$a = \text{apple}$ and $b = \text{bananah}$ [sic] (apples can be added to apples, but you can't add apples and bananahs together). [Year 7 student, pre-questionnaire, school G]

I have never done it before but I guessed that if there was an ' a ' you add 1, ' b ' you add 2, etc. [Year 7 student, pre-questionnaire, school G]

It's just added in to make it look harder but it's easy. [Year 8 student, pre-questionnaire, school B]

Nothing, anything. [Year 8 student, pre-questionnaire, school B]

I think the a and b mean ... [Year 8, pre-questionnaire, school C]

² The student used the questions $6a + 2a$ and $12b - 2b$ to illustrate her answer.

As these examples show, this category contained a great variety of responses. Since this category also included all those who left the question blank, the proportion of students who appeared not to understand letters in any kind of numerical way may be an over-estimate. A blank line might not mean that the student did not know how to interpret letters at all, merely that they had decided not to answer the question for some reason. Just as the number of students in the 'algebraic' category was probably an over-estimate, so the 'other' category probably included students who did have some understanding of letters.

Table 7 shows the detailed results for the students as a whole, and for the Phase II survey students as a separate group. Responses are to the question asking students what they thought the letters used in algebraic questions might mean. Figures in brackets are the actual *numbers* of students in each category; figures not in brackets are the *proportions* of students in each category, expressed as a percentage of the number of students completing each questionnaire³. The top part of the table gives the results for all students, but does not include the delayed post-questionnaires, since school *G* students did not do these. The lower part of the table gives the results across all three questionnaires for the Phase II survey students (schools *A* to *D* only), so that information for the delayed post-questionnaires can also be included.

³ Since the pilot study students were not asked this question, there are 30 fewer students represented in this table than in the previous one.

Table 7: Students' responses to a direct question, asking how they interpreted letters

		<i>Pre-questionnaires</i>	<i>Immediate post-questionnaires</i>	<i>Delayed post-questionnaires</i>
<i>All students</i>	<i>Total</i>	(384) 100%	(379) 100%	
	<i>'algebraic'</i>	(154) 40%	(210) 55%	
	<i>'numeric'</i>	(38) 10%	(47) 12%	
	<i>'other'</i>	(192) 50%	(122) 32%	
<i>Survey students only</i>	<i>Total</i>	(279) 100%	(272) 100%	(283) 100%
	<i>'algebraic'</i>	(120) 43%	(142) 52%	(160) 57%
	<i>'numeric'</i>	(18) 6%	(28) 10%	(26) 9%
	<i>'other'</i>	(141) 51%	(102) 38%	(97) 34%

This table indicates that some significant changes did occur in students' ability to answer such a question satisfactorily, with more students aware that letters represent numbers.

Considering the results for all the students, the proportion of students who could state that letters represent numbers or some variant on this ('algebraic') increased from 40% to 55% across the teaching period, and the proportion of students who showed no awareness that letters stand for numbers at all ('other') decreased from 50% to 32%.

The survey students were also considered separately, both so that the delayed post-questionnaires could be taken into account, and so that the degree to which the results of the case study students from school *G* were generalisable to the other schools could be assessed. The results for the survey students were initially less impressive, but were very similar by the time of the delayed post-questionnaires. The proportion of students in the 'algebraic' category rose from 43% to 52% by the immediate post-questionnaires, and then to 57% by the delayed post-questionnaires. The proportion of students with a totally non-numerical understanding ('other') had dropped from 51% to 38% by the immediate post-questionnaires, and to 34% by the delayed post-questionnaires. These findings suggest that on this indicator, at least, results from the case studies and the survey were not greatly dissimilar, once the delayed post-questionnaires results were taken into account.

It would appear that many students were better able to articulate an understanding of what letters used in an algebraic context mean after they had completed their graphic calculator modules. Although this is a good result, the figures do raise some concerns. It is worrying that so many students did not understand letters to be numbers at all (that is, were in the 'other' category). Most of the survey students (all but the 36 Year 6 students) had studied some algebra before, yet half of them had not grasped the fact that letters stand for numbers. By the time of the delayed post-questionnaires, this had dropped to about a third, which indicates progress was made, but that there was still work to be done in this area. Furthermore, this change may not imply that the students were able to make use of this improved verbal competence when working with algebraic expressions, as discussed in the introduction to this section.

6.3.2 Level of algebraic questions students can successfully answer

During 1974-79, the Concepts in Secondary Mathematics and Science (CSMS) research programme investigated the performance of students in the 11 to 16 years age range on eleven secondary school mathematics topics. Their assessment of students' performance in algebra was reported by Küchemann (1981). Since 1981, his framework has become a standard for assessments of the level at which students can understand and work with algebraic expressions (e.g. Booth, 1984; Graham and Thomas, 1998). In his study, Küchemann formulated a hierarchical evaluation of students' interpretations of letters. Throughout this chapter, this evaluation is used to estimate the level of understanding of letters and expressions which students actually demonstrated when working on standard algebraic questions, as opposed to their professed understanding in answer to a verbal question about the meaning of letters in algebra.

Before proceeding with this analysis, however, I wish to clarify how I interpret Küchemann's levels here. Küchemann made tentative connections between his levels and the levels a student might be expected to have reached according to Piaget's developmental levels; for instance, level 1 corresponded to "[b]elow late concrete", whereas level 4 corresponded to "[l]ate-formal" (Küchemann, 1981: 117). I have rejected Piaget's developmental framework as a basis for understanding students' capacity to learn algebraic concepts in this thesis (section 2.3.1), but Küchemann's levels do provide a way to track any progress made by students in the specific areas of interpretation of letters and competence in working with expressions. Whereas Piaget considered his levels to be *necessary* stages a child passes through on the way to maturity, Küchemann's framework does not require students to show earlier levels of interpretation of letters before they reach higher levels (cf. Ávalos, 1996), but provide a means of describing the stage that a student has reached in her/his thinking about variables. It is quite possible for a student to miss all the level 1 and 2 stages in Küchemann's hierarchy, and start with a level 3 understanding of a letter as a specific unknown, for instance. Indeed, it was an intention, in using the graphic calculator model of a variable, to enable this to happen for students who were new to algebra.

Küchemann's analysis

From the performance of some three thousand students in Years 8 to 11 (aged 13 to 15) from schools across the UK, Küchemann and his colleagues were able to formulate a hierarchy of levels of understanding. This hierarchy consisted of six different categories of interpretation of a letter at four different levels of understanding (Küchemann, 1981: 104).

'Letter evaluated', 'letter not used', and 'letter used as an object'⁴, which were all deemed to be ways of avoiding working with the letters, were put at level 1 or 2. 'Letter used as a specific unknown' and 'letter used as a generalised number' were at least level 3, and possibly level 4. 'Letter used as a variable' was put at level 4. Using these levels, Küchemann classified each question in his tests according to the interpretation of a letter the question required and its structural complexity.

It is not possible to be certain that a student who answers a high level question correctly does in fact have that level of understanding, as Küchemann acknowledged (p111).

However, a student who can answer one higher level question correctly, despite not really having that level of understanding, is unlikely to sustain correct answers at that level.

Küchemann considered a level had been achieved by a student if at least two thirds (approximately) of the questions at that level were answered correctly, and this criterion was also used where possible in the analyses in this chapter. In the Year 7 case study, however, there was only one question at each level⁵.

Many of the questions used in my questionnaires were taken from those used by Küchemann and his colleagues, so that direct comparisons could be made. Others that were similar were given a level on his framework; those that were not similar were ignored

⁴ For example, a means 'apple' and $6a$ means 'six apples'. This is unhelpful, because although it stresses the difference between different variables so that students see why $6a + 2b$ cannot be simplified ('you can't add apples and bananas'), it does not help students to see that $6a$ is the product of two numbers.

⁵ In my role as the class teacher, I wished to avoid making students who were new to the school, and had little if any previous experience of algebra, unduly worried about what might be in store for them.

for the purpose of these analyses. Examples⁶ of questions used for the analyses in this chapter are given below⁷.

Level 1 (letter can be interpreted as an object, evaluated, or ignored)

$$2a + 5a = \quad \quad \quad [all\ questionnaires]$$

$$a + b = 57, a + b + 2 = \quad \quad \quad [Year\ 8G\ study]$$

Neither of these questions requires students to understand anything about the use of letters, or to operate directly on them, to obtain a correct answer. The first can be answered by simply adding the coefficients, then putting the letter back. Alternatively, it can be conceptualised as two apples plus five apples, which clearly gives seven apples. The second requires the student merely to observe that the required answer will be 2 more than 57.

Level 2 (as level 1, but structurally more difficult, with perhaps some lack of closure in the answer)

$$12b - 2b = \quad \quad \quad [Year\ 7G\ and\ Year\ 6-8\ studies]$$

$$4a + 3b + 2a = \quad \quad \quad [Year\ 6-8\ study]$$

Although the first question here appears similar to the first level 1 example above, students found questions involving subtraction more difficult, so such questions were considered structurally more difficult in this analysis. In Küchemann's study, questions like $4a + 3b + 2a$ were put at level 2, whereas questions like $4a + 3b - 2a$ were assigned to level 3. A question like $4a + 3b + 2a$ requires a student to accept that the correct answer is $6a + 3b$, that is, to accept lack of closure. It was clear from the questionnaires where this

⁶ Each questionnaire in a particular study carried questions of the same structure and difficulty, but used different letters and numbers so that students would not feel they were doing the same questionnaire as before. For example, $6a + 2a$ on the pre-questionnaire would become $3b + 9b$ on the immediate post-questionnaire, and so on.

⁷ Full details of all the questions used for this analysis can be seen in Annex IV.

type of question was used that many students of the age group involved in this research were unhappy with an answer like this, preferring to give a single number or term, such as 9, $9ab$, or $9c$.

Level 3 (letter needs to be interpreted as a specific unknown, and some lack of closure in the answers needs to be accepted)

Add 4 onto $7n$

[Year 9G and Year 8G studies]

$5a - 2b + 7a =$

[Year 8G and Year 6-8 studies]

These questions require the student to accept both lack of closure in the answer, and some understanding of the meaning of the letters involved. Students not able to cope at this level tended to add the coefficients and either ignore the letters, or put them together in some way. The first question above was frequently 'simplified' to 11 or $11n$, the second to 10, $10ab$ or $10c$ (or even 12, $12ab$, or $12c$). Küchemann put $5a - 2b + 7a$ at a higher level than $4a + 3b + 2a$ because the latter could be answered by combining numbers of objects (such as apples and bananas), whereas the former is more difficult to interpret in this way:

3 apples take away one banana makes little immediate sense (unless there already are some bananas), nor does 3 a 's take away one b unless b is thought of as a number.
(Küchemann, 1981: 107)

Level 4 (as level 3, but structurally more difficult, requiring interpretation of a letter as a generalised number or a variable)

Multiply $3 + b$ by 9

[Year 9G study]

Is $a + b + c = a + p + c$ true:
always/sometimes/never?
Choose one of these alternatives,
and explain your answer.

[Year 8G study]

The first question here is structurally more difficult than the first exemplar of level 3, according to Küchemann (1981: 109), since students have to realise that both terms in the expression $3 + b$ need to be multiplied by the 9. The second question requires students to understand that b and p may sometimes have the same value, despite being represented by different letters. This would indicate that students were beginning to interpret letters as

generalised numbers, rather than specific numbers. Students who had not made this shift in understanding were likely to state that the two expressions could never be the same, as b and p were necessarily different numbers.

Questions from all the questionnaires used in the Phase I case studies (school G) and the Phase II survey (schools A , B , C and D) were, where possible, allocated a level on Küchemann's framework. In the Year 7 case study, there was only one question at each level on each questionnaire. Level 4 questions were only used in the pilot study (Year 9G) and the follow-up case study (Year 8G); questions used in the Year 7G case study and the Year 6-8 survey were at most level 3.

Using Küchemann's framework to assess the case study and survey students' competence in working with algebraic expressions

Once the questions on all the questionnaires had been allocated a level on Küchemann's framework (or omitted from this analysis), the numbers of students achieving a given level were determined, and details of this analysis are shown in Table 8. The top part of this table represents the results of *all* the students, and the bottom part represents the results of the *survey* students. Figures shown in brackets are the *numbers* of students who achieved at least the given level, by answering correctly at least two thirds (approximately) of the questions at this level. Figures not in brackets are the *proportions* of students achieving at least a given level, expressed as a percentage of the total number of students completing each questionnaire. (Students who failed to answer correctly at least two thirds of the questions at level 1 were put at level 0 in subsequent analyses).

Table 8: Proportions (and numbers) of students achieving a given level on Küchemann's framework

		<i>Pre-questionnaires</i>	<i>Immediate post-questionnaires</i>	<i>Delayed post-questionnaires</i>
<i>All students</i>	<i>Total</i>	(414) 100%	(409) 100%	
	<i>At least level 1</i>	(327) 79%	(338) 83%	
	<i>At least level 2</i>	(231) 56%	(279) 68%	
	<i>At least level 3</i>	(109) 26%	(126) 31%	
<i>Survey students only</i>	<i>Total</i>	(279) 100%	(272) 100%	(283) 100%
	<i>At least level 1</i>	(234) 84%	(228) 84%	(247) 87%
	<i>At least level 2</i>	(177) 63%	(178) 65%	(182) 64%
	<i>At least level 3</i>	(79) 28%	(74) 27%	(57) 20%

This table suggests that the students as a whole made a little progress on this indicator, but that the picture for the survey students was less good. Considering the students as a whole first, some improvements in the levels achieved on Küchemann's framework can be seen in Table 8. Initially 21% of the students were unable to answer correctly even questions at level 1, but this decreased to 17% by the time the students did the immediate post-questionnaires. 26% of the students were initially at level 3 or higher, and this increased to 31% by the immediate post-questionnaires. The most noticeable change, however, was in the proportion of students reaching level 2 successfully, with a rise from 56% of the students to 68%.

Looking at the results for the survey students, the change in the proportion of students reaching at least level 1 was similar by the time of the delayed post-questionnaires.

However, the impressive rise in the proportion of students reaching level 2 did not occur, and there was actually a decrease in the proportion of students able to answer questions at level 3 correctly by the time of the delayed post-questionnaire. These results are considered further in the next section.

6.3.3 Comparison between students' responses to a direct question about the meaning of letters, and their level on Küchemann's framework

In section 6.3.1, students' interpretation of letters in answer to a specific question was considered; in section 6.3.2 the level at which students were actually operating in answering algebraic questions was discussed. The next question to answer is the degree to which these are related. Does the ability to answer successfully the question: "What do the letters in an algebraic question stand for?" mean that students can operate at some minimum level on Küchemann's framework, or is there no connection between the two? This is now explored further.

Levels on Küchemann's framework achieved by students sub-divided by their interpretation of letters

As might be expected, a relationship was found between students' direct responses about what letters mean, and the levels they reached on Küchemann's framework. To explore this relationship, students were divided into two groups according to their responses to the direct question about their understanding of letters⁸. The 'algebraic' category, as in the earlier analysis, comprises students whose response indicated that they understood letters to be numbers in general, or at least specific unknowns. The 'non-algebraic' category here includes students previously put into either the 'numeric' or the 'other' category: that is, all those students whose understanding does not appear to be at the level of a specific unknown. The level on Küchemann's framework reached by students in each of these subsets is then considered.

⁸ Again, the Year 9G students are excluded from this analysis, since they were not asked a question asking them what the letters in algebraic questions mean.

The level on Küchemann's framework which should correspond to an understanding that letters represent numbers in general is level 3. This is the level which Küchemann identified as the stage where students begin to operate on letters as at least specific unknowns, rather than avoiding them in some way, treating them as objects or replacing them with specific values. Table 9 shows that not all students with an 'algebraic' understanding of letters reached level 3 on Küchemann's framework, and not all students with a 'non-algebraic' interpretation of letters were at a lower level than this. That some students are able to answer a direct question about the meaning of letters successfully without being able to answer higher level questions is perhaps not surprising. What is less expected, perhaps, is that some students whose understanding was 'non-algebraic' reached at least level 3 on Küchemann's framework and others with an 'algebraic' interpretation did not even reach level 1. These students show distinct differences between their professed interpretations of letters, and the level at which they could actually answer algebraic questions successfully.

The details of this analysis are shown in Table 9. Sub-division by interpretation of letters was done first. Figures in brackets are then the *numbers* of students in each subset who achieved the given level on Küchemann's framework. Figures not in brackets are the *proportions* of the students in each subset achieving the given level, expressed as a percentage of the number of students *in that subset completing each questionnaire*.

Table 9: Comparison of students' verbal interpretation of letters with the levels they achieved on Küchemann's framework

	'non-algebraic' interpretation of letters		'algebraic' interpretation of letters	
	Pre-questionnaires	Immediate post-questionnaires	Pre-questionnaires	Immediate post-questionnaires
Total	(230) 100%	(169) 100%	(154) 100%	(210) 100%
At least level 1	(159) 69%	(120) 71%	(139) 90%	(188) 90%
At least level 2	(96) 42%	(88) 52%	(108) 70%	(163) 78%
At least level 3	(34) 15%	(24) 14%	(55) 36%	(74) 35%

Table 9, like Table 7, shows clearly the increase in the *number* of students able to interpret a letter 'algebraically'. There were decreases at all levels between the pre-questionnaires and the immediate post-questionnaires in the numbers of students in the 'non-algebraic' subset, with corresponding increases in the numbers of students who knew that letters represent numbers.

Within both subsets, the most impressive increases were in the *proportions* of students reaching at least level 2. For students in the 'non-algebraic' category, that proportion rose from 42% to 52%, and for those in the 'algebraic' category it rose from 70% to 78%. All other changes in proportions are very minor.

'Anomalies' in the comparison of students' verbal understanding and levels achieved on Küchemann's framework

There were 99 students whose interpretation of letters on the pre-questionnaires was 'algebraic', but who did not show they could answer questions successfully at level 3 on Küchemann's hierarchy. This can be deduced from Table 9, since there were 154 students altogether in the 'algebraic' subset, of whom only 55 achieved at least level 3. These 99 students represent 64% of those in the 'algebraic' subset, and 26% of the student body as a whole. This means that nearly two thirds of students who were aware that letters represent numbers were not able to work at a level which Küchemann claimed was equivalent in understanding. This is perhaps not that difficult to explain: it is much easier to learn verbal definitions than to make use of the knowledge which such definitions encapsulate, as Vygotsky pointed out so frequently (eg. Vygotsky, 1986: 158).

By the time of the immediate post-questionnaires, there were 136 students⁹ able to state that letters represent numbers (and thus in the 'algebraic' group), who could not answer level 3 questions successfully, corresponding to 65% of that group, or 36% of the entire student body. This finding suggests that the proportion of those able to answer appropriately the question about their understanding of letters who were unable to answer questions at level 3 had not changed since the pre-questionnaires, but that more of the students were in that position. This again supports Vygotsky's view that it is not verbal competence which counts in establishing a new concept, but that such verbal competence is only the beginning of genuine learning: "The development of scientific concepts *begins with the verbal definition.*" (Vygotsky, 1987: 168, original italics).

Before the graphic calculator modules began, there was also a group of 34 students¹⁰, who were in the 'non-algebraic' subset, but who were able to answer successfully questions at Küchemann's level 3 (all but one from the survey schools). These students represent 15% of those in the 'non-algebraic' subset and 9% of the entire student body. On the immediate post-questionnaires, 24 students¹¹ (all from the survey schools) failed to answer satisfactorily a question about the meaning of letters, but were able to answer successfully questions at level 3. They represented 14% of those in the 'non-algebraic' subset at this point, or 6% of the entire student body.

⁹See Table 9: there were 210 students in the 'algebraic' group for these questionnaires, of whom 74 achieved level 3.

¹⁰ See Table 9: 'non-algebraic' subset, 34 students achieved level 3.

¹¹ See Table 9: 'non-algebraic' subset, 24 students achieved level 3.

This is a far more surprising finding, and one which requires further research. Their presence tends to suggest that concept formation, and the development of accompanying skills, is a complex business, with some students able to answer algebraic questions at a level beyond their verbal competence. Such students perhaps realise intuitively what to do, without actually verbalising what this might mean about the letters involved.

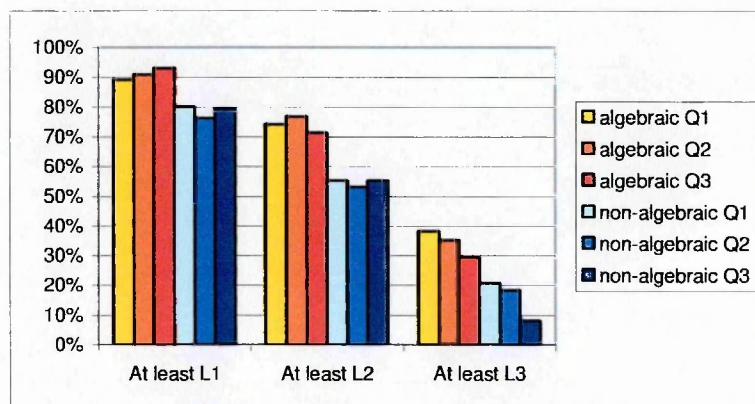
Alternatively, they may have been rote taught, so that they can simplify expressions without really understanding what they are doing. It would be interesting to see how they fare at algebra after another year or two.

The existence of these two groups of students would seem to indicate that knowing that letters represent numbers is neither a necessary nor sufficient condition for students to be able to answer questions at Küchemann's level 3. One conclusion to draw from this, is that although Küchemann's level 3 corresponds to an understanding that letters are specific unknowns at least, working at this level demands far more of students than just this realisation. It may also be that the other kinds of learning required for students to work successfully at level 3 are more important than simply realising that letters are used as generalised or even specific numbers.

A closer look at the survey students

Since there did not appear to be much change in the levels reached on Küchemann's framework by the survey schools, it seemed worth investigating if there was any difference in the levels achieved by students giving an 'algebraic' response to the question about their interpretation of letters and those giving a 'non-algebraic' response.

Figure 15: Proportions of students achieving at least each of Küchemann's levels (expressed as a percentage of the students in the 'algebraic' and 'non-algebraic' categories respectively)



The yellow, orange and red bars in Figure 15 show the proportion of the students giving an 'algebraic' interpretation of a letter achieving at least each of levels 1, 2 and 3, while the pale, mid and dark blue bars show the proportions of those with a 'non-algebraic' interpretation achieving each level. The absolute number of students in the 'algebraic' category increased across the questionnaires, while the absolute number in the 'non-algebraic' category decreased (Table 7, p202).

Of the students in the 'algebraic' category, around 90% reached at least level 1, compared to around 80% of those in the 'non-algebraic' category, while over 70% of those in the 'algebraic' category reached at least level 2, compared to around 55% of those in the 'non-algebraic' category. However, the proportion reaching at least level 3 decreased across the three questionnaires for both groups, from 38% to 35% to 29% in the case of those in the 'algebraic' category, and from 21% to 18% to 8% in the case of those in the 'non-algebraic' group.

Clearly a higher proportion of students in the 'algebraic' group reached each of the levels. The proportions of students reaching at least levels 1 and 2 remained fairly constant in both categories, but the drop at level 3 occurred in both categories of students. Whatever the cause of this drop, it does not therefore appear to be related to students' interpretation of a letter. From the data available, it is not possible to probe this finding more deeply. It is quite possible that the drop at level 3 is a statistical effect caused by the increase in the population of the 'algebraic' group and decrease in that of the 'non-algebraic' group. Other explanations include the possibility that these students were suffering from 'questionnaire fatigue', or that their performance was affected by other factors which were not apparent from the data.

6.3.4 Summary of this section

This is a complex section in which I have attempted to explore students' understanding of letters, comparing it with the level they can achieve on Küchemann's framework. In section 6.3.1, students' responses to a direct question asking how they interpreted the letters used in algebraic expressions were analysed. Overall, the students made progress in this, with a greater proportion able to answer satisfactorily after the graphic calculator modules than before. This finding appeared to be consistent across the whole student body and the survey students alone.

In section 6.3.2, Küchemann's analysis was used to determine the level at which students could actually answer algebraic questions correctly. His analysis depends on the interpretation of a letter required by a question and its structural complexity, and so is another way of looking at student understanding. Here the students as a whole appeared to make a small amount of progress, but this was not shown by the survey students.

In section 6.3.3, these two analyses were combined, to see if students' verbal interpretation of a letter was related to the level they could reach on Küchemann's hierarchy. There was found to be some relationship between the two, but there were two groups whose results appeared anomalous. These included those whose interpretation of a letter was 'algebraic', but who had yet to achieve Küchemann's level 3, and those whose interpretation of a letter was not 'algebraic', but who had achieved Küchemann's level 3. Using the subdivision of the survey students into 'algebraic' and 'non-algebraic' subsets, their results on Küchemann's hierarchy were re-examined. It was found that students in both categories showed a similar pattern in Küchemann's levels, with little change in those reaching at least levels 1 and 2, and a drop in those reaching at least level 3.

In this section, students' understanding of letters and how it is connected to the level of questions they can successfully answer has been considered. In the next section, students' proceptual understanding is examined, together with the implications this has for the level they can achieve on Küchemann's hierarchy.

6.4 STUDENTS' PROCEPTUAL UNDERSTANDING OF EXPRESSIONS

Proceptual understanding (Tall and Thomas, 1991) is a measure of a student's ability to see an expression like $2x + 3$ both as a set of operations on a variable, x , and as a mathematical entity in its own right which can be operated on in the same way that x can. Students' facility in interpreting expressions flexibly as operations on a single variable, or as mathematical entities, according to context is an important indicator of their capacity to see how expressions are constructed and hence how to operate on them (section 2.6.5). This is necessary for proceeding beyond the most basic of levels in algebra, and so progress in this

aspect of students' understanding was investigated to see if there had been any change during the period the students worked with the graphic calculators.

Students' proceptual understanding can be illustrated by responses from the survey students to a question asking them:

How did you work out your answer to question 5 [$4a + 3b + 2a =$]?

On the pre-questionnaires, many students gave responses like these:

I worked it out by adding 4, 3 and 2 together and then adding on an a . [*Answer given, 9a; Year 7 student, school B*]

Added the numbers up together. [*Answer given, 9a; Year 8 student, school A*]

By the immediate post-questionnaires [the corresponding question was $6c + 2a + 3c =$], these two students' responses were respectively:

I added $6c$ and $3c$ together and then placed the $2a$ on the end. [*Answer given, $9c2a$; Year 7 student, school B*]

I added the two c numbers to the $2a$. [*Answer given, $9c + 2a$; Year 8 student, school A*]

It was clear that at least some of the students were viewing expressions more holistically or proceptually than they had done initially, and less as collections of numbers to be operated on, with letters as mere appendages.

An example of a question requiring proceptual understanding (from the Year 8G case study) is $2a/2a$. If a student is to understand that $2a/2a$ is equal to 1, s/he needs to see the two ' $2a$ ' terms proceptually, that is, as single mathematical entities, so that the whole expression is perceived as something divided by itself. Students whose understanding was not yet proceptual did not see the ' $2a$ ' terms as single entities, and so did not see the overall structure of the question. Instead, they tended to work on the coefficients first, getting a value of 1 or 0, onto which they might or might not tag an ' a '. An answer of 1

could be an indication of proceptual thinking; an answer of $1a$, 0 or $0a$ is a clear indication of its lack.

This one example is not enough to give complete certainty about the degree of proceptual understanding a student has, since a student could get a value of 1 by cancelling the two '2's, and completely ignoring the letters. However many students whose thinking was at this stage tended either to divide 2 by 2 to get 0 rather than 1, or to put an 'a' onto the number. This is equivalent to answering a question like $6a + 2a$ by adding the '6' and the '2', then putting the 'a' back on to get $8a$ ¹². This lack of proceptual thinking is not obvious in the case of $6a + 2a$, but is in other questions.

Questions¹³ from each of the questionnaires used in the case studies and survey were analysed according to the degree of proceptual understanding required in order to answer them successfully. Other examples of questions used to test proceptual understanding were:

$$5a - 2b + 7a =$$

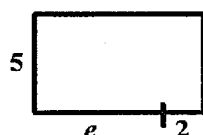
[Year 6-8 and Year 8 study]

Add 4 onto $7n$

[Year 9G and Year 8 study]

Which of the following expressions do you think is correct for the area of this rectangle? Tick every one you think is correct.

[Year 9G study]



¹² This corresponds to one of Küchemann's level 1 forms of understanding of a letter: that a student operates on the numbers ignoring the letters, at least until the final answer is displayed.

¹³ Details of the questions used can be seen in Annex V.

(Options: $5 \times e + 2$, $5 \times (e + 2)$, $10e$, $5 \times e^2$, $5(e + 2)$, $e + 2 \times 5$, none correct/other answer)

The Year 7G questionnaires did not have any questions which could be used to assess proceptual understanding, so these students were omitted from this analysis.

Taking the students as a whole, on the pre-questionnaires 51% of them showed evidence of proceptual thinking, which increased to 54% on the immediate post-questionnaires. The proportions of students from the Phase II survey showing proceptual thinking showed little change, remaining around 49-51% on all three questionnaires.

Such figures conceal considerable variation, however. Some students clearly came to this work with a good understanding of what a letter meant, and how it should be used, whereas others had no idea about using letters in an algebraic way. To probe such differences, the degree to which students' thought was proceptual was investigated according to their initial level on Küchemann's framework. This level was chosen rather than age, because any one age group could contain as much variation in understanding and attainment as the whole group, whereas a student's initial level on Küchemann's framework gives a baseline for their ability to work with algebraic expressions. It was chosen in preference to their response to the direct question asking about their interpretation of letters, because it reflects the level at which a student can actually work, rather than a purely verbal understanding.

Students were assigned a level on Küchemann's hierarchy for their pre-questionnaires, and this level was then used throughout this analysis, so that a student assigned to level 1 on the pre-questionnaire was considered as a level 1 student thereafter, regardless of any progress made on subsequent questionnaires. The degree to which they showed proceptual

thinking was then assessed for all the questionnaires, and compared from the pre-questionnaire to the post-questionnaire(s).

Details of this analysis are shown in Table 10. Figures shown in brackets are the *number* of students at a given level on Küchemann's framework showing proceptual thinking *as a proportion of the total number of students at that level*. Figures not in brackets are the same proportions converted to a percentage. So, for instance, none of the 46 (0/46) students at level 0¹⁴ on the pre-questionnaires showed proceptual thinking, whereas six of the 41 (6/41) of these students completing the immediate post-questionnaires showed proceptual thinking (the other 5 students did not complete the immediate post-questionnaires). Because the Year 7 students from school G did not have any questions suitable for showing proceptual thinking, only the Year 8 and 9 students from school G are included in this analysis (57 students on the pre-questionnaires).

Table 10: Proportion of students at a given level on Küchemann's framework in the pre-questionnaires who showed proceptual thinking

	Level on Küchemann's framework	Proportion of students showing proceptual thinking		
		Pre-questionnaires	Immediate post-questionnaires	Delayed post-questionnaires
All students	Total	(172/336) 51%	(172/319) 54%	
	Level 0	(0/46) 0%	(6/41) 17%	
	Level 1	(4/67) 6%	(9/62) 16%	
	Level 2	(64/115) 57%	(63/106) 62%	
	Level 3 or 4	(104/108) 96%	(94/100) 94%	
Survey students only	Total	(138/279) 49%	(128/253) 51%	(126/258) 49%
	Level 0	(0/45) 0%	(5/40) 13%	(4/38) 11%
	Level 1	(2/57) 4%	(6/52) 12%	(9/53) 17%
	Level 2	(57/98) 58%	(51/90) 57%	(48/91) 53%
	Level 3	(79/79) 100%	(66/71) 93%	(65/76) 86%

¹⁴ Those who failed to answer at least two thirds, approximately, of the level 1 questions on the pre-questionnaire successfully.

It can be seen from Table 10, that proceptual thinking appears to be closely linked to a student's level on Küchemann's hierarchy. The proportion of students showing proceptual thinking is very low for students operating at levels 0 and 1, but some of them were able to show a degree of proceptual thinking after the graphic calculator modules. More than half of the students able to answer questions at level 2 successfully showed proceptual thinking, and virtually all the students able to operate at level 3 or above showed proceptual thinking. The only improvements in proportions of students showing proceptual thinking occurred in those students originally on the lowest levels according to Küchemann; students originally at a higher level showed drops in the proportions showing proceptual thinking. This links with some decreases in performance shown by these students in other analyses described in this chapter.

It is inevitable that there will be a fair degree of agreement shown in this analysis between the level on Küchemann's framework achieved by students and the proportion showing proceptual thinking, since the questions at the higher levels according to Küchemann are also those which require proceptual understanding, whereas lower level questions do not. It would be interesting to conduct further research on this, using different questions for determining students' level on Küchemann's framework and for determining their proceptual understanding, to see if the agreement shown here is an artefact of the questions used, or whether it is a real connection between the two measures.

6.5 PROGRESS IN BASIC ALGEBRAIC SKILLS

A third measure of students' progress during the period they worked with the graphic calculators is their ability to answer questions of the type met in traditional algebra teaching. This is also a way of finding out if the graphic calculator method of teaching

algebra compares successfully with other methods as regards the development of such skills. Practising such traditional skills was not a basic objective of the teaching materials used, but it was hoped that enabling students to understand better how and why letters are used in algebra would contribute to the development of such skills (cf. Graham and Thomas, 1998; Graham and Thomas, 2000a). If students are able to answer skills-based questions more fluently as well as showing an increase in their conceptual understanding, then the graphic calculator model can certainly be deemed useful.

Progress in basic algebra skills was found to follow the same kind of pattern as previous analyses. Students with a low starting point on Küchemann's framework made good progress, while students starting from a higher level appeared to stand still, or even, in some cases, to regress.

This analysis included all the questions used on the questionnaires, not all of which were used in the other analyses discussed in this chapter. Whereas the questions used to determine students' levels on Küchemann's framework and those used to determine their conceptual understanding were often the same questions, all the questions on the questionnaires were used for the basic skills scores, many of which could not be used in other analyses, so this result is based on a wider sample of questions. This means that this result is more independent of the other analyses used (although still not entirely independent). The questions are not rated by the level of understanding they require or their structural complexity, as with the measures dependent on Küchemann's analysis.

Development of basic skills was measured by looking at students' scores on all the algebraic questions on each questionnaire. All the previous analyses have used *numbers or*

proportions of students in a given category. For this analysis, however, each questionnaire was scored according to the number of algebraic questions it contained, and each student was given a mark, expressed as the *percentage of questions answered correctly*. These marks were then averaged across each questionnaire for all the students completing it, and a mean correct score obtained for each questionnaire.

On the pre-questionnaires, the students considered as a single group showed a mean correct score of 47%, which improved to 54% on the immediate post-questionnaires. This increase is statistically highly significant¹⁵ ($p = 4.24 \times 10^{-5}$), and very unlikely to have occurred by chance. For the survey students taken as a single group, the students' mean score was 49% on the pre-questionnaires, which improved to 52% on the immediate post-questionnaires, but dropped back to 51% on the delayed post-questionnaires. None of these changes is statistically significant.

Again, the students' results were investigated further by grouping students according to their initial level on Küchemann's framework, so that variations in the performances of different groups of students could be ascertained. Similar results were obtained to those in the previous analyses: the students in the lowest achieving groups made excellent progress, while those in the highest achieving group showed a small decrease in their mean score. Details of this analysis are shown in Table 11, where students are again sub-divided according to their level in the pre-questionnaires on Küchemann's framework (note that these figures give *mean scores* rather than proportions of students).

¹⁵ All results used for calculations of statistical significance can be seen in Annex VI.

Table 11: Comparison of mean scores for each questionnaire, students differentiated according to their level in the pre-questionnaires on Küchemann's framework

	Level on Küchemann's framework	Mean scores		
		Pre-questionnaires	Immediate post-questionnaires	Delayed post-questionnaires
All students	Level 0	15%	41%	
	Level 1	43%	45%	
	Level 2	53%	55%	
	Level 3 or higher	71%	74%	
Survey students	Level 0	21%	34%	35%
	Level 1	35%	38%	40%
	Level 2	53%	54%	51%
	Level 3	70%	71%	68%

The most remarkable feature of this table is the gain in mean score shown by the students initially at level 0 both in the whole group of students and in the survey. Looking at the student body as a whole first, the increase in the mean score for those at level 0, from 15% on the pre-questionnaires to 41% on their immediate post-questionnaires is statistically so unlikely, that a p value cannot be obtained ($|z| = 8.45$). A similar result is seen among the level 0 survey students, whose mean score rose from 21% to 34%. The probability of this increase happening by chance is $p = 3.53 \times 10^{-7}$. These values both indicate that it is highly unlikely that these students would have obtained the increased scores that they did without some intervention, and is thus an indication that the graphic calculator modules that they studied helped them to develop markedly better basic algebra skills.

None of the other changes shown in Table 11 is statistically significant. In general, the students showed increases in their scores, which they maintained through to the delayed post-questionnaires. Exceptions to this are the level 2 and 3 students from the survey schools, who showed a small decrease in mean score between the immediate post-questionnaires and the delayed post-questionnaires.

6.6 SUMMARY AND CONCLUSIONS

In this chapter, analyses of the questionnaires given to the participating students were used to determine how successful the graphic calculator model and teaching materials had been in helping students to understand letters and expressions better, and also to see if this affected their basic algebra skills. Given the nature of the case studies and the survey carried out, it is not possible to be certain that the progress noted is due solely to the graphic calculator modules that the students studied. The purpose of this research was to find out what would happen if graphic calculators were used to provide a model of a variable and to support students' investigational work in ordinary classes, rather than in controlled settings. In everyday classroom lessons, there are always many variables, and no attempt was made to control for these.

Three different analyses were used to see if the questionnaire data support the contention that the graphic calculator provides an everyday model for a variable which can interact with the abstract model taught in school, resulting in new conceptual learning, as Vygotsky maintained. The first was students' understanding of letters, as demonstrated both verbally and in their responses to typical algebra questions. The second was the degree to which students showed proceptual thinking, and the third was that of the skill they showed in answering typical algebra questions.

The first analysis was divided into two different aspects. Initially, students' responses to a straightforward question asking them what they thought the letters in the algebraic questions meant were analysed. It was found that about half the students knew that letters represent numbers, but that half had no idea what the letters meant. Even after the graphic calculator modules, with their emphasis on putting numbers into stores, about a third of all

the students were unable to give a satisfactory answer to such a question. This finding alone is significant for practitioners. It is no wonder that students find algebra hard, when so many of them fail to grasp the fundamental fact that letters stand for numbers. It seems likely that the graphic calculator model did help some students, but others still needed to have the model reinforced. Vygotsky emphasised that verbal learning was but the beginning of conceptual learning, and thought neither 'everyday' nor 'scientific' knowledge enough on its own (Vygotsky, 1986: 194).

Students' understanding was also investigated by looking at the level of questions they could answer successfully. To determine the level of a question, Küchemann's (1981) hierarchical framework was used, in which questions are classified both by the understanding of letters required and their structural difficulty. Overall, the students showed some improvement here, but this was found to derive from the case study students with those in the survey showing little change, or even a drop in level.

These two aspects of understanding were then considered together. Students were subdivided into 'non-algebraic' and 'algebraic' subsets, according to whether they could state that letters stand for numbers, or at least a specific unknown. Looking at the levels on Küchemann's framework students in each subset achieved, it was found that many were at a level which accorded with their verbal understanding, but that a minority either had a verbal understanding ahead of their ability to work on a question, or vice versa. Students with a verbal understanding ahead of the level at which they could actually answer questions exemplify Vygotsky's (*Collected Works*, Vol III, van der Veer and Valsiner, 1994) claim that verbal knowledge is but the beginning of concept formation. The much smaller group of students whose demonstrated ability to work on questions appeared to be

in advance of their verbal understanding indicates that this whole area is more complicated than that simple statement would suggest. Further research in this area would be useful, to see if this result is simply due to inconsistencies in students' understanding and ability to work on questions at this stage, or whether there is another explanation.

Progress made by the students in their proceptual understanding was analysed, with students differentiated according to their initial level on Küchemann's framework. It was found that those students whose initial level was very low made some progress, whereas students at the highest levels slipped back a little. Proceptual understanding appeared to relate to initial level on Küchemann's framework, but this may well be an artefact, caused by the use of many of the same questions in the analyses.

Students' progress in basic skills showed that the lowest achieving students made spectacular progress, whereas other students' progress was less remarkable. Perhaps the excellent progress of the students with lowest previous success at algebra may be attributed to the graphic calculator model and teaching method combining the 'everyday' and the 'scientific', enabling them to develop some understanding of basic algebra.

In Chapters 4 to 6, the findings of the case studies and survey have been considered from various perspectives. In Chapter 4, the effects of the graphic calculator as a cognitive tool were discussed, then in Chapter 5, vignettes from the classroom were analysed to show examples where cognitive change occurred and examples of occasions where it failed to occur. Finally, in this chapter, the progress students made in understanding the interpretation of letters and the construction of expressions, and their skill in using these was discussed. In the next chapter, students' errors and misconceptions are described,

together with the effect on these misconceptions of the graphic calculator model of a variable.

CHAPTER 7 MISCONCEPTIONS

7.1 INTRODUCTION

During the analyses conducted to determine how much progress students had made during the classroom studies carried out in this research, it became obvious that many students showed a range of misconceptions. Students who had not yet really started algebra did not start their study of it as ‘blank slates’, and very few of these students (less than 10%) had no misconceptions at all. Some of these misconceptions showed themselves to be remarkably persistent, still observable in students some two years older, and with nearly two years more algebra teaching. Vygotsky’s view, that: “[a]ny learning a child encounters in school always has a previous history” (1978: 84) would suggest that students already have many ideas that they use in trying to construct meaning for what they do in algebra: there is no such thing as starting from scratch. Instead, the everyday conceptions children have already formed have to be considered (Vygotsky, 1987), and where necessary children have to be given the means to reconstruct how they link these concepts with what they learn in school.

Students’ failures at algebra (section 2.6.7) and the literature on the occurrence of misconceptions when students make the transfer from arithmetic to algebra (sections 2.6.3, 2.6.8) have been discussed in Chapter 2: *Review of the literature*. Some researchers have proposed the existence of a cognitive ‘gap’ (Herscovics and Linchevski, 1994; Linchevski and Herscovics, 1996) or cognitive ‘obstacles’ (Booth, 1988; Filloy and Rojano, 1989). However, Tall (1989) suggested that “our curriculum, designed to present ideas in their logically simplest form, may actually cause cognitive obstacles” (p89), rather than their

being inherent. Sutherland (1991) agreed, arguing that the apparent existence of such 'gaps' or 'obstacles' may be caused by the use of Piagetian theory, which contends that if students are unable to cope with the demands of algebra it is because they have not yet reached the stage of formal operations.

A Vygotskian perspective supports this possibility. The nature of students' learning depends on the tools they use (as discussed at length in sections 2.5 and 4.2), so students' misconceptions may be a result of the instruction process and the tools used, rather than of the existence of a pre-determined 'gap' or 'obstacle'. This is the view taken in this thesis. Consequently, it is worth discussing further how using the graphic calculator in the way described earlier (section 4.4) might help students deal with misconceptions. It is argued here that the unit of two students and a graphic calculator can enable students to extend their ZPD, through dialogue supported by the calculator which provides language and a forum for investigating ideas (section 4.3). This environment encourages students to try out their ideas, to see which are correct and which are not. Examples where this occurred were given in Chapter 5: *Evidence of cognitive change*.

In this chapter, students' misconceptions are analysed in detail, and the effect of using the graphic calculator on these is considered. In section 7.2, major and minor misconceptions are described. Here major misconceptions are defined as those observed in more than 10% of the Year 6 and 7G students, who were new to algebra, while minor misconceptions are those observed in between 5% and 10% of these students. Section 7.3 is a discussion of the types of misconceptions students showed before and after their graphic calculator modules and the proportion of students in which these were observed. In this section, the incidence of the various types of misconception is considered for each year group of

students, rather than by students' level on Küchemann's framework, as in the previous chapter. I decided to do this because misconceptions were spread throughout the student body, and I wanted to see if the length of time for which students had been studying algebra made any difference to the incidence of misconceptions. In section 7.4, misconceptions deemed to be a result of the teaching method are discussed. Finally, section 7.5 is a summary of the chapter.

7.2 MISCONCEPTIONS OBSERVED IN STUDENTS NEW TO ALGEBRA

The pre-questionnaires of the Year 6 students from school B^1 and the Year 7 students from school G^2 were analysed for misconceptions the students brought with them before they started formal work in algebra. Although many of these students had done preliminary work designed to lead up to learning algebra including some use of letters for numbers, few had done very much formal algebra at the point that they did these questionnaires. It is not possible to pinpoint a time at which students start learning algebraic concepts, since all concepts are grounded in the totality of a child's experience. However, in the UK educational system the change from Year 6 to Year 7 marks the end of the primary phase and the beginning of the secondary phase, and is the point at which a more formal approach to mathematics teaching often occurs. I hoped therefore that the Year 6 students, who were just a couple of months from this transition, and the Year 7G students, who were a couple of months past this transition, would give an indication of the algebraic misconceptions students have already formed at this significant point in their education.

Students' answers to questions on their pre-questionnaires were coded for different categories of misconceptions shown. Any misconception shown by at least 10% of the Year 6 and 7G students taken together, equivalent to three or more students in a class of 30, was deemed a major misconception. Any misconception shown by between 5% and 10% of the students was deemed a minor misconception. Misconceptions observed in less than 5% of the students were not considered further, apart from one particular instance which appeared to be teacher related, and which is discussed further in Section 7.4. Using these criteria, five major misconceptions were observed in these students and one minor misconception.

7.2.1 Major misconceptions observed

These were those found in the questionnaires of more than 10% of the Year 6 and Year 7G students taken together.

2a means a '2' and an 'a', and therefore equals $2 + a$

This misconception was only found in the questionnaires of the Year 6 students, since there were no questions where it could have been observed in those of the Year 7G students. However, it was shown by 83% of the Year 6 students, and was the most prominent misconception shown by older students in the Year 6-8 survey also. The Year 6 students were asked to indicate any expression they thought might be the same as $2a$ from choices which included $2 + a$. A further question asked students to mark any answer they agreed with for bc , given that $b = 3$ and $c = 5$, and here 8 was one of the choices. The

¹ The non-selective middle school.

² The selective girls' grammar school.

essence of this misconception, therefore, is to add two symbols which are written next to each other without any operation indicated.

For students who have not been introduced to the algebraic convention that such symbols should be multiplied, this is quite reasonable. In arithmetic, students would have learnt that $2\frac{1}{2}$ means $2 + \frac{1}{2}$ and 34 means $30 + 4$, so, extending these ideas, $2a$ could mean $2 + a$ (Matz, 1980). However, it seems unlikely that students actually think this through consciously in this way. Students who can answer $12b - 2b$ correctly often give an answer of 2 to the apparently structurally similar problem of $2b - b$. This appears to be because they see $2b$ as a pair of symbols, a '2' and a 'b', rather than proceptually as $2b$: if you have a '2' and a 'b' and you take away the 'b' you are left with the '2' (Gage, 2002a). $12b - 2b$, on the other hand, is dealt with by subtracting the 2 from the 12, and then tacking the b back on the end, giving an apparently correct answer. If this argument is correct, then $2a$ is perceived as a '2' and an 'a', and bc as a 'b' and a 'c'. This interpretation is supported by the two Year 9G students, who chose options of '5 and y' and $5 + y$ as correct interpretations of $5y$, and the discussion with Sally reported below (and also discussed in 1.2).

Sally was the Year 9G girl whose observation that $2x - x = 2$ started this research; it is clear from the discussion quoted below that she did not really have a concept of terms like $2x$. This extract is from the first lunchtime session Sally and I held, having decided she needed extra help. We were going through a revision sheet which the class had done a couple of weeks earlier. In trying to solve $5 - 3x = 23$, Sally had written:

$$\begin{array}{l} x \xrightarrow{\times 3} x \times 3 \xrightarrow{-5} x \times 3 - 5 = 23 \\ 7 \xleftarrow{\div 3} 28 \xleftarrow{+5} 23 \end{array}$$

which we started to discuss. I suggested she start this again from scratch. This time, she added the 3 to the 23 to get 26, then tried to divide by the 5. Sally clearly had no concept of the $-3x$ as a proceptual unit.

A little later, we moved on to consider $6 + x = 7 + 2x$. The conversation continued:

- | | | |
|-------------------|--|---|
| JG ³ : | So what we do, is we use this idea of balancing ...
On the left hand side if you take away x , what do you get left with? | <i>I reminded Sally briefly of work we had done earlier in class on equations where the idea of balancing was used.</i> |
| Sally: | Just the plus ... | <i>$6 + x$ less the x leaves just $+$. It is apparent how little comprehension Sally had of what was going on⁴.</i> |
| JG: | Not just the plus sign ... | |
| Sally: | ... and 6 ...
... | <i>$6 + x$ less the x leaves $6 +$...</i> |
| JG: | But we have to do it to both these sides, and if we take away an x on this side, what do we get? | <i>Turning to consider the right hand side of the equation ...</i> |
| Sally: | 2.
... | <i>$7 + 2x - x = 2$</i> |
| JG: | No, $2x$ means 2 times x , two lots of x , if you like. | <i>I tried to give $2x$ some meaning ...</i> |
| Sally: | OK. | |
| JG: | If I've got two lots of x , and I take one of them away ... | <i>Then I tried to give the operation $2x - x$ some meaning ...</i> |
| Sally: | You've got one of them ... | |

Sally was perfectly capable of understanding that if you have two lots of something and you take one lot away, you are left with one lot of it. The problem is that she (and other students like her) do not conceptualise $2x$ as two lots of x , but see it is a '2' and an ' x '.

³ JG is me.

⁴ No doubt, she was not alone in this. Although she appeared weak by comparison with her class, she obtained a Level 5 pass at the end of Key Stage 3 SATs, which is the target grade at this stage.

“I added them all together and added an ‘a’”

This response of a Year 6 student to the question: “How did you work out your answer to $4a + 3b + 2a$ ” illustrates the next major misconception shown by the Year 6 and 7G students. The answer he gave was $9a$, which he justified as above. Students showing this misconception simply work with the coefficients, and then perhaps put a letter or letters on afterwards. Other examples of responses from Year 6 students⁵ who ignored the letters were:

I just added the numbers up and put $a - b$ after because I didn’t know what to do as there were two different letters. [answer of $9a - b$]

I added all the number up and add to it the next number [sic] of the alphabet. [answer of $7c$]

Add the numbers together and put a c because there is an a and a b therefore a add b equals c . [answer of $9c$]

At least 61% of the Year 6 group and 33% of the Year 7G group answered one or more questions by simply ignoring the letters, and operating on the coefficients. The true proportion could be much greater than this, since it is not possible to know if the student who gets an answer of $8a$ to $6a + 2a$ has in fact added the ‘6’ and the ‘2’, and then just put an ‘a’ on the end. Ignoring the letters either completely or until the last minute is one of the low level ways of working with letters identified by Küchemann (1981) (see 6.3.2), who observed this misconception in at least 37% of the 14 year-old students in his study (p106).

⁵ Year 7G students were not asked to explain the reasoning behind their answers in questions like this.

This is an example of a misconception where a strategy which works at one level fails at a higher level. Answering $6a + 2a$ in this way is fine, but it causes problems very soon, as can be seen in answers to this question:

Add 4 onto $n + 43$ [Question from the Year 8G and 9G questionnaires]

Such answers included 47 and $47n$. Another example of this approach causing problems can be seen in this extract from a discussion Nicky and Karen had during the first lesson of the Year 8G follow-up case study:

Nicky: Question 1, $3x = 12$. So what we do is 12 divided by 3?
Karen: How does the x come in?

$“6a + 2a = 82”$

The third misconception considered here is called ‘code’, and was observed in some of the examples described in section 5.2.1 also. Many students interpreted a term such as $2a$ as the digits of a two-digit number, or as a code in which letters stand for single digit numbers. This is familiar to them from simple codes and mathematical puzzles where letters are used to represent digits. In the pre-questionnaires, 35% of the Year 6 and the Year 7G students demonstrated this misconception. The Year 6 student, whose answer is quoted above, justified her answer by putting: “I think $a = 1$ and $b = 3$ ”. She had obviously worked out $6a + 2a$ to mean $61 + 21$ to obtain her answer of 82; her answer to $5a - 2b + 7a$ was 99 (obtained by evaluating $51 - 23 + 71$). Similarly, a Year 7G student gave answers of 84, 100 and 85 to $6a + 2a$, $12b - 2b$ and $6a + 2b$. She wrote that the letters in these questions were to be interpreted as $a = 2$ and $b = 3$. Given values of $b = 3$ and $c = 5$, many of the Year 6 students chose options for bc of 35 or even 23 (from $b = 2$ and $c = 3$, despite their being given specific values in the question).

This also proved to be a tempting way of solving an equation like $2j - 6 = 18$. In a question asking them to explain how they had reached their answers for this and similar equations, two students wrote:

$j = 4$ because $24 - 6 = 18$ [Year 6 student, solving $2j - 6 = 18$]

Add twelve to 24 = 36 so the $x = 6$. [Year 7G student, solving $3x - 12 = 24$, original underlining]

This is an example of a misconception obtained by extending a practice which works in codes and puzzles of the type:

Three A's

In the multiplication problem below, each letter represents a different digit:

$$\begin{array}{r} A\ S \\ \times \ A \\ \hline M\ A\ N \end{array}$$

Which of the ten digits does A represent? (Summers, 1968: 6)

An example of this type of thinking is seen in this excerpt, recorded by two students during the first lesson in the Year 7G case study, and already discussed in section 5.2.1. This is quoted again to show how the graphic calculator has the potential to enable students to challenge this particular misconception.

- | | | |
|------------|--|--|
| Abigail: | [Number] three is 2A.
Two and four must be 24, I think. | <i>They were given $A = 4$ to use. This is a perfect example of the 'code' misconception.</i> |
| Charlotte: | Well, I'm not sure, so I'll just write it anyway! | |
| ... | | |
| Charlotte: | Three, 2A. | <i>A little later on, they checked the question with the graphic calculator. Graphic calculator key sequence ...</i> |
| Abigail: | 2A, so 2 ALPHA A. 2 ALPHA A ... | |
| Charlotte: | ... equals 8. | <i>... confirming that their answer was not right. Charlotte giggled, then there was a long pause.</i> |
| Abigail: | Whoops! We're wrong ... | |

Other students showed this misconception in their written work. In her first piece of homework, Fran put $3C = 33$ and $2C - D = 22$, given that $C = 3$ and $D = 1$; on the other hand, she put $3 \times C = 9$. She obviously did not understand $3C$ to be $3 \times C$. Similarly, Briony wrote $CDCD = 3131$, although she also wrote $2C + 2D = 8$. In an interview when the module was finished, Briony said her father had agreed with her that $CDCD = 3131$. Later in the same homework, using values of $C = 2.2$ and $D = 0.2$, Fran put $3 \times C = 4.4$ (which is actually $2 \times C$), $3C = 6.6$ (which is correct) and $2C - D = 4.2$ (also correct). Briony, however, put $CDCD = 2.20.22.20.2$, which is at least consistent!

$a = 1, b = 2, c = 3, \dots$

A common misconception (23% of the Year 6 and 7G students showed this on their pre-questionnaires) was to equate a with 1, b with 2, c with 3, and so on. Given an expression to simplify, then asked what the letters a and b in that question referred to, two students wrote:

$$4a + 3b + 2a = 12a$$

I just worked it out by assuming that $a = 1$ and $b = 2$ and just added it together and put the answer in a 's. [Year 6 student]

$$6a + 2b = 6 + 4 = 10$$

$$a = 1, b = 2 \text{ [Year 7G student]}$$

In both these questions, the values of 1 and 2 have been substituted for the a and b . In other examples, they were used as digits (the 'code' misconception discussed above). This misconception is caused by extending the idea that a is the *first* letter of the alphabet, b is the *second*, and so on, to give values for the letters on this basis. Again, this is often used in simple codes and puzzles.

Questions in which such alphabetic substitution could be seen included questions like those above, and the multi-choice questions on the Year 6 questionnaires, where one of the options for $2a$ was 21, and 23 was an option for bc . Students choosing these options demonstrated this particular misconception. Instances were even seen of equations 'solved' by this means, so that, for example, $j = 10$ was given as a solution to $2j - 6 = 18$.

Substitution of numbers for letters

This category was used to indicate instances when students substituted numbers for letters where there was no need to do this or it was inappropriate to do so. One example is those students who evaluated expressions like $4a + 3b + 2a$ by substituting numbers of their choice for the letters, giving a purely numerical answer. Explaining what they thought the letters meant, two students wrote:

I think a means $\times 1$ and b means $\times 2$. [Year 6 student, who obtained an answer of 12 to the above question]

I think (not sure) that a stands for 2 and b for 1? [Year 7G student, explaining an answer of 14 to $6a + 2b$]

This occurred in the pre-questionnaires of 15% of the Year 6 and Year 7G students, and is an example of a misconception which starts by helping the students, by giving them a way of understanding algebraic terms, but soon restricts their understanding.

7.2.2 Minor misconceptions observed

The major misconceptions described in the previous section were observed in more than 10% of the questionnaires completed by students in the Year 6 and Year 7G groups. One minor misconception was observed, demonstrated by between 5% and 10% of the students from these groups, which was an instance of algebraic rules being extended inappropriately to numbers. It seems unlikely that this particular misconception is amongst those which the children would have brought with them from their previous experience, however.

Given the nature of this misconception, it seems more likely to have occurred because the students misunderstood what their teachers told them about algebraic terms. This misconception involved errors in dealing with terms like $24pq$, with equivalents such as $2 \times 4 \times p \times q$ and $42pq$ being selected. This is considered further in Section 7.4, which concerns misconceptions which arose during the teaching process.

Although all the Year 6 and most of the Year 7G students had done some algebra before, this was probably their first introduction to formal algebra. It therefore seems reasonable to make the assumption that the major misconceptions shown by these students are those students bring with them from their 'everyday' experience (Vygotsky, 1986). This is supported by the nature of these misconceptions: that two symbols written next to each other should be interpreted as one plus the other; that it makes sense to work with familiar numbers then tack meaningless letters on at the end; that a letter stands for a single digit; that $a = 1$, $b = 2$, and so on; that numbers should be substituted for letters in order to give an answer which makes sense (that is, it is numerical).

7.3 INCIDENCE OF MISCONCEPTIONS AMONG ALL THE STUDENTS

The previous section considered the misconceptions shown by the youngest groups of students, those more or less at the start of their formal algebra teaching, in order to give a picture of the misconceptions students bring with them from their previous school work and their everyday lives. In this section, all the misconceptions shown by all the students who took part in the various classroom studies are considered before and after their work with the graphic calculator, in order to see if the same pattern of misconceptions existed as with the younger students, and whether working with the graphic calculator had any impact on those shown.

It was found that the same misconceptions as discussed above in section 7.2 emerged, plus one extra one. Five misconceptions were identified as major misconceptions in the previous section. These, together with names given to identify them, were:

- ‘sum’: interpreting e.g. $2a$ as $2 + a$
- ‘letters’: ignoring the letters and working on the coefficients
- ‘code’: interpreting e.g. $2a$ as twenty something
- ‘alphabetic’: assuming $a = 1$, $b = 2$, $c = 3$, and so on
- ‘substitution’: substituting numbers for letters inappropriately

The additional misconception found was another of the low level interpretations of letters mentioned by Küchemann (1981), the identification of letters with objects, which he reported observing in over 20% of 14 year-olds (p107). This misconception was demonstrated by 5% of the Year 7G students (four people) on their pre-questionnaires but it was not shown at all by the Year 6 students, reducing its incidence to 4% of this group as a whole, and thus below the threshold chosen for considering a misconception as of even minor significance. Since the incidence of this misconception is so low among the youngest students, it is considered in section 7.4 with the other errors deemed to have arisen during the teaching process. This section focuses on the overall incidence of misconceptions which students appear to hold prior to any formal teaching about algebra.

On the pre-questionnaires taken as a whole across all the year groups and schools participating in all the classroom studies (407 students⁶), the proportion of students

⁶ 414 students took part altogether, but 7 of the Year 7G students did not answer any of the algebraic questions, so their misconceptions could not be considered.

showing no observable misconceptions at all was 18%. This proportion rose with age, as might be expected, apart from the Year 8 group in schools A, B, and C. Table 12 shows how this proportion varied with the age of the students.

Table 12: Proportions of students showing no misconceptions in the pre-questionnaires

<i>Year group</i>	<i>Average age at time of classroom study</i>	<i>Number of students</i>	<i>Proportion of these showing no misconceptions</i>
Year 6 (School B)	Just over 11 years	36	3%
Year 7G (School G)	Just over 11 ½ years	71	13%
Year 7 (Schools B and D)	Just over 12 years	124	21%
Year 8G (School G)	Just over 12 ½ years	27	26%
Year 8 (Schools A, B and C)	Just over 13 years	119	14%
Year 9G (School G)	About 14 ¼ years	30	43%

While some of these figures cannot be taken as representative of wider populations of students, since only small numbers are involved, the table does show that over half of these particular 14 year-olds, who were at the end of Key Stage 3 in a selective school, still had observable misconceptions. This proportion rose to over four in five of the Year 8 students from the three non-selective schools. This finding is in agreement with Graham's (1998) remarks:

Unfortunately, largely because of student embarrassment, the true extent of their [students'] confusion is never fully revealed – a state of affairs which can seriously undermine their confidence and damage their capacity to grasp new, more challenging ideas. (p22)

Apart from one particular subgroup, the students did make some progress in this area.

Overall, 18% of the students showed no observable misconceptions on the pre-questionnaires, which rose to 21% on the immediate post-questionnaires. Considering just the survey students, who also did a delayed post-questionnaire, 16% of students showed no observable misconceptions on the pre-questionnaires which rose to 19% on the immediate post-questionnaires, then fell to 15% on the delayed post-questionnaires.

Table 13: Changes in the incidence of the five major misconceptions between the pre-questionnaires and the immediate post-questionnaires

Misconception	Year 6	Year 7G	Year 7	Year 8G	Year 8	Year 9G
'sum'	- 12%	-- ⁷	- 9%	--	- 4%	- 3%
'letters'	+ 8%	+ 4%	- 24%	- 13%	- 3%	+ 30%
'code'	- 16%	- 9%	- 24%	- 23%	- 30%	--
'alphabetic'	- 8%	- 5%	+ 10%	0% ⁸	- 3%	- 3% ⁹
'substitution'	- 3%	+ 6%	+ 2%	+ 7%	+ 5%	- 3% ¹⁰

The negative figures in Table 13 denote decreases in the incidence of a given misconception in a year group, and in fact, most of the changes were decreases (shaded pink). This is useful supporting evidence for the beneficial effects of the graphic calculator in aiding students to remediate their misconceptions, as well as providing them with the means to practise their skills. Exceptions to this are shown by the particularly high increase in 'letters' among the Year 9G students, and the smaller increase in the Year 6 and 7G students. There is also an increase in 'alphabetic' thinking among the Year 7 students, and several small increases in the incidence of 'substitution' in other groups. However, it seems clear that the graphic calculator model, and the graphic calculator method of working, helped students who had previously interpreted terms as 'codes'. This conclusion is evidenced by the big decreases in this misconception across all groups. Many students were also helped to understand that pairs of symbols standing next to each other are not to be understood as sums, and that a does not always have the value 1, and so on. The helpfulness of the graphic calculator is also backed up by the big drop in the proportion of Year 7 and 8G students who ignored letters in favour of numbers.

⁷ '--' indicates that there was no question on the given questionnaire which could be used to test for a particular misconception.

⁸ 0% incidence both before and after.

⁹ From 3.3% to 0%.

¹⁰ From 3.3% to 0%.

A question like $2a + 5b$, or similar, was used to test how many students ignored the letters and worked on the coefficients amongst the Year 9G students, and this question was also used on the Year 7G questionnaires, where a much smaller increase was observed. On the pre-questionnaire, many of the Year 9G students made comments to the effect that:

This equation [*sic*] cannot be simplified as they are different letters.

None of these [*the options given*] are equivalent, because you still only have $2a$ and $5b$.

Students, who answered this correctly on the Year 9G pre-questionnaire, then wrote comments on their post-questionnaires like:

$3 \times 6 = 18$. $3d + 6c = 18dc$. d or c haven't been cancelled out.

You add the $6 + 3$. $3d + 6c = 9dc$.

It is not clear from the data why these students should have done this question correctly the first time, with explanations which made good sense, but then have written answers showing a much less clear understanding the second time. Perhaps initially the students remembered what they had been taught previously about adding terms involving different letters, but, after two weeks of working on various algebraic expressions and equations, were much more primed to find an 'answer'. However, there is nothing in the data to indicate if this is so or not.

Many improvements were maintained through to the delayed post-questionnaires, as can be seen from the figures for the survey students in Table 14:

Table 14: Changes in the incidence of the five major misconceptions between the pre-questionnaires and the delayed post-questionnaires in the Year 6-8 survey

Misconception	Year 6	Year 7	Year 8
'sum'	- 8%	- 3%	-5%
'letters'	+ 5%	+ 4%	- 6%
'code'	- 3%	- 15%	- 14%
'alphabetic'	- 6%	- 1%	- 3%
'substitution'	+ 4%	+ 2%	+ 1%

The changes here are much less dramatic, but there is still a good decrease in the incidence of 'code' thinking in the Year 7 and Year 8 students. The incidence of the 'sum', 'code' and 'alphabetic' misconceptions also decreased over the six week period across all three year groups. However, 'letters' and 'substitution' showed small gains, particularly with the younger students. Overall, this table supports the claim that the graphic calculator method of working can help students to remediate their misconceptions. It would be foolish to anticipate eliminating these particularly resistant misconceptions in just a few lessons, however, without considerable further work targeted precisely at exposing and helping students change such errors in their thinking.

7.4 MISCONCEPTIONS OBSERVED AS A RESULT OF THE TEACHING PROCESS

All the misconceptions discussed so far were observed in more than 10% of the Year 6 and Year 7G students in their first formal algebra work. Two other misconceptions were also present at this stage, but to a much lesser extent. Both of these were observed to be more significant after the teaching modules, and they are therefore deemed to have arisen as a result of the teaching process. Of course, this is not to say that any of the teachers were at any point deliberately teaching errors to their students. However, it seems likely that some of the students misunderstood some of the things said to them, and hence developed these misconceptions.

7.4.1 'Product' errors

The first such misconception relates to products, and two variants of this were observed. In one variant, the digits in a number were split up, so that 12 became 1×2 ('product separation'), and in the other, the digits were reversed, so that 12 became 21 ('product

reversal'). Students justified the first of these with the convention that symbols written side by side in algebra are to be multiplied: if $2a$ means $2 \times a$, then perhaps $12a$ could mean $1 \times 2 \times a$. The justification for the second used this together with the commutative rule: since $2a = 2 \times a = a \times 2 = a2$, then $12pq = 1 \times 2 \times p \times q = 2 \times 1 \times q \times p = 21qp$. These students, like those described by Booth (1984; 1988) and Lee and Wheeler¹¹ (1987; 1989) (section 2.6.6), clearly thought there was no reason to expect the laws of arithmetic to hold in algebra. Many students also do not really understand what the '=' sign means, seeing nothing wrong with writing strings like $3 + 5 = 8 \times 2 = 16$.

I discussed one of these variants with two of the Year 7G students in their interviews at the end of the module. I wrote down $24st = 2 \times 4 \times s \times t$ and asked them if they thought this could be correct. Nisha agreed that this was correct, but Renelle was less sure, drawing attention to the 2×4 . Although she could see something was wrong, Nisha could not, repeating that what I had written was fine. I therefore suggested they substitute some numbers and evaluate the expression, which they did. Having worked out what the two expressions would be for $s = 2$ and $t = 3$, both girls then agreed that something was wrong, and Renelle reiterated that 2×4 was 8, not 24. This exchange confirmed what Nisha had written in the homework she did after their first lesson with the graphic calculators:

$$24st$$

$$2 \times 4 \times s \times t$$

$$24 \times s \times t$$

$$20dc$$

$$20 \times dc$$

$$2 \times 0 \times d \times c$$

¹¹ Lee and Wheeler noted that a third of the students involved in a study they carried out accepted an answer of $20 = 4$ (e.g. 1989, p45f), seeing nothing wrong with this statement, as it was algebra not arithmetic.

'Product' errors were also observed in the Year 8G case study during the second videotaped class discussion, as the following exchange shows:

- | | |
|--|---|
| <p>Teacher: $36bc$. Vote on which are the same. $cb49$... anyone? Why do you think that's the same?</p> <p>Student: Because when they're next to each other they're times.</p> <p>Teacher: When they're next to each other you multiply?</p> <p>Student: Yeah, and 4 times 9 is 36.</p> <p>Teacher: 4 times 9 is 36 ...</p> <p>Student: You do ... yeah, and then you do that times b and then times c.</p> | <p><i>The class were asked which of various given expressions were equivalent to $36bc$. A handful of students felt that $cb49$ was acceptable, and the teacher asked one of them why she thought that.</i></p> <p><i>So $cb49 = 4 \times 9 \times b \times c$</i></p> |
|--|---|

The incidence of 'product' errors increased from the pre-questionnaires to the post-questionnaires, as shown in Table 15 below, suggesting that this misconception arises as a direct result of the teaching process. It is common to hear a teacher telling students that when symbols are written next to each other in algebra they are to be multiplied, and that it does not matter which way round you put symbols or numbers when multiplying. The discussion above shows how misunderstanding remarks like these can all too easily lead students to develop misconceptions.

In Table 15, the incidence of the two variants of the 'product' error is given for the pre-questionnaires, the immediate post-questionnaires, and the delayed post-questionnaires, where these were done. The figures given are the percentages of the students from each year group demonstrating the error on each questionnaire¹².

¹² Year 9G did not have any questions which would have revealed either of these misconceptions.

Table 15: Incidence of ‘product’ errors across the questionnaires

	‘product separation’ ¹³	‘product reversal’ ¹⁴
Year 6	6% → 23% → 9%	6% → 6% → 13%
Year 7G	Not tested → 21% ¹⁵	
Year 7	8% → 17% → 12%	3% → 2% → 7%
Year 8G	15% → 25%	4% → 7%
Year 8	3% → 21% → 19%	4% → 8% → 6%

‘Product separation’ occurred to a much greater extent than ‘product reversal’, except with the Year 6 students. The trend for both misconceptions was for their incidence to increase across all the year groups between the pre-questionnaires and the post-questionnaires, suggesting that these are indeed ‘taught’ misconceptions, arising from students not hearing properly what is said, or not fully understanding it.

7.4.2 Interpreting letters as objects

The teacher of the school C Year 8 group commented on the questionnaire which she completed after she had finished teaching the graphic calculator module:

I was intrigued by the number of pupils saying a and b represented apples and bananas, etc. After questionnaire 3 I asked who had taught them this – apparently it was one particular feeder school, hence one group of children with this idea.

This particular misconception was not shown at all by the Year 6 students and by only four of the Year 7G students, two of whom wrote that algebraic letters meant:

a = apple and b = bananah [*sic*] (apples can be added to apples, but you can’t add apples and bananahs together. [Year 7G student, pre-questionnaire])

Apples and bananas, apes and bats. [Year 7G student, pre-questionnaire]

¹³ This means separating out the digits in a product, e.g. $12ab = 1 \times 2 \times a \times b$.

¹⁴ This means reversing the digits in a product, e.g. $12ab = 21ab$.

¹⁵ Neither of these misconceptions was tested for on the Year 7G pre-questionnaire. On the post-questionnaire, options were given which did not differentiate the two misconceptions.

It was more apparent in some of the responses of older students, however. On her pre-questionnaire, a Year 8G student¹⁶ answered the question:

$$a + b + c = a + p + c$$

Is this true: always/sometimes/never

with:

Never ... because for example, you could not make apples, bananas and carrots equal apples, pears and carrots. [*Year 8G, pre-questionnaire*]

After she had completed the graphic calculator work, she did the same question again, this time answering:

Sometimes ... because you can have different answers so you have different numbers. [*Year 8G, post-questionnaire*]

Küchemann also pointed out that letters can be treated as objects even if they are not interpreted as specific objects, like apples and bananas. Students who view expressions like $2a$ as a '2' and an 'a' are doing this. A question where this could be detected was:

$$2b - b = 2 \quad [\textit{Year 8G, both questionnaires, several students}]$$

As was discussed above in section 7.2.1, the thinking here seems to be that if you have a '2' and a 'b' and you take the b away, you are left with the 2: b is conceptualised as one of a pair of objects rather than as a number multiplied by 2.

Another way in which the identification of letters with objects was observed was in a method used by some of the Year 8G students to solve equations where the unknown appeared on both sides of the equation. They would write xx for $2x$, which worked well for them when the coefficients were whole numbers, but failed them when the coefficients were not whole numbers, as the following excerpt from the Year 8G case study shows.

¹⁶ This student was new to the school in Year 8, so was not involved in the Year 7G graphic calculator work.

The students were given a set of equations to solve, amongst them $7x + 1 = 2x - 3$.

Rebecca's comments are given below, together with the corresponding written work she produced:

Rebecca:	I'm going to start by drawing seven x 's ... and then add 1, and then equals two little x 's, and then minus 3 ...	$xxxxxxx + 1 = xx - 3$
	... and then you cross out the two little x 's and two little x 's of the first side, and then you end up with five little x 's add 1 equals minus 3.	$xxxxx + 1 = -3$

Rebecca's commentary and her written work both show her conceptualisation of the terms $7x$ and $2x$, as a collection of 'little x 's', that is, as objects. This worked perfectly well for the first part of this question, leading her to the simplification $5x + 1 = -3$. She and her partner, Fran, then went astray by subtracting 1 from -3 to get -2 , rather than -4 . There is nothing in what follows on the audiotape to suggest how they eventually got to (almost) the right answer, but Rebecca's written work finally read:

$$7x + 1 = 2x - 3$$

$$xxxxx + 1 = -3$$

$$5x + 4 = 0$$

$$4 \div 5 = 0.8 \text{ [there is another sign error here]}$$

The class teacher was asked about this method of doing equations in her interview at the end of teaching this module. Using $3x + 2 = 2x - 5$ as an example, she explained:

Well, I suppose it's maybe from the balance¹⁷, and me saying that on your balance you've got three x 's and you've got two, so they've drawn the three x 's, as you would on a balance, and on this side of your balance you've got your two x 's, so if we want to remove these objects ... OK, they've got a value, we don't know their value, but they're still something you can call an x ... When I saw them doing that, I said that's

¹⁷ She was referring to the teaching method for equations known as 'balancing', in which students are encouraged to model their equation on a pair of balance pans which must be kept in balance by doing the same operation on both.

OK, but you must write down $x + x + x \dots$ [They don't] but that's sort of laziness, I think. I mean they probably started off by saying, OK, we'll write $x + x + x \dots$ and quite soon, they think we can do this more quickly by not bothering to ...

This sounded perfectly reasonable, both as an explanation of what the students were doing, and as a way of coping with equations of this type. However, Rebecca and Fran found that this method was of no help with equations like $1.5 - 0.5x = 3.5 + 1.5x$ (which was structurally equivalent to the equation they had just solved). I would argue that what the students were doing in abbreviating $x + x$ to xx is not about laziness, but about a deficient understanding of what $2x$ means.

This became apparent from Rebecca and Fran's discussion about what $1.5x$ means. $2x$ can be represented by xx and then worked with, but $1.5x$ is much harder to represent in this way, and representing $-0.5x$ on a balance using 'little x 's' is surely beyond the point at which this model is useful, as can be seen from the following transcript:

Rebecca:	We've got 1.5 on both sides ... so that's 1.5 minus something and 1.5 plus something ... so if we have 1.5 plus 3.5 that makes 5 ...	$1.5 - 0.5x = 3.5 + 1.5x$ <i>Rebecca started by ignoring the x's, and just working with the coefficients. She began by trying to simplify the right hand side of the equation to 5.</i>
Fran:	1.5 minus 0.5 is 1 ...	<i>Fran did the same on the left ...</i>
Rebecca:	... so we've got 5 and 1 ...	<i>... leading them to simplify the left hand side to 1 and the right hand side to 5.</i>
Fran:	0.5 and one x ...	<i>Fran read $0.5x$ as 0.5 and an x, confirming that she saw x as a separate object.</i>
Rebecca:	1.5 minus 0.5 equals 1 ...	<i>Rebecca was still ignoring the x, working on the coefficients only.</i>
Fran:	... but it's not 0.5, it's $0.5x$, so it's 0.5 times x .	<i>At this point, Fran saw that so far they had been misinterpreting these terms.</i>
Rebecca:	Good point. 1.5 equals 0.5 times something, and 3.5 add 1.5 times something, so ... 1.5 minus 0.5 is 1, and then 3.5 add 1.5 is 5, and then times something to make them the same ...	<i>Rebecca agreed and ... then went back to simplifying using coefficients only!</i>

At this stage, neither Rebecca nor Fran had managed to find a way into the question: their previous method, using ‘little x ’s’ had not even been considered. This demonstrates its inadequacy that it cannot be used for all equations of this structure, but only for those where there are small, integer coefficients. As a result of playing with the coefficients, Rebecca decided that 10 could be significant, basically since both 5 and 1 are factors of it. Fran then jumped at something that sounded as if it might yield a value for x , suggesting that as 10 is 5×2 then x might be 2. They both went round in circles a little more, playing around with 10, 5 and 2, but eventually decided to leave this question for the time being.

The degree of confusion these students showed on this question contrasts with the relative assurance they showed in working with the structurally equivalent equation $7x + 1 = 2x - 3$. The presence of the non-integer coefficients showed up the inadequacy of their method, and their inability to see the expressions proceptually¹⁸, rather than as collections of ‘little x ’s’.

Table 16 gives the figures for the incidence of the various manifestations of the ‘object’ misconception. Figures are for the pre-questionnaires, the immediate post-questionnaires, and the delayed post-questionnaires where these were done¹⁹.

¹⁸ A proceptual understanding of, e.g. $7x$, is as a term which can be considered as an object in its own right, as well as the result of a multiplication, or the sum of seven ‘little x ’s’.

¹⁹ There were no questions on the Year 9G questionnaires which allowed ‘object’ thinking to be seen; school G students did not do delayed post-questionnaires.

Table 16: Proportion of students on each questionnaire who showed evidence of ‘object’ thinking

Year group	% of students
Year 6	0% → 0% → 0%
Year 7G	6% → 3%
Year 7	7% → 9% → 8%
Year 8G	26% → 21%
Year 8	7% → 11% → 7%

Unlike the ‘product’ misconception, the ‘object’ misconception does not show a rise in all the year groups: indeed, it falls in most cases. Surprisingly, the Year 8G students showed a particularly high incidence of this misconception, although the reason for this is not clear from the data. However, if the Year 6 and Year 7G students can be assumed to give an indication of misconceptions students hold prior to being taught algebra, then it is not present to any great extent at this point. So-called ‘fruit salad’ algebra may be used by some teachers as a way to help students understand why terms with different letters in them cannot be added together. Using ‘little x ’s’ to help with certain equations may well be taught, or at least tolerated, as part of the ‘balance’ method of solving equations. Hence the ‘object’ misconception is also considered to be produced in students through the teaching process.

7.5 SUMMARY AND CONCLUSIONS

The data collected in this research study suggest that students have far more misconceptions than is often realised, and that their incidence is considerably greater than many teachers may suppose. Even at the age of 14, misconceptions were observed in the errors made by more than half of the students, and younger students almost all showed misconceptions.

Misconceptions held by the Year 6 and 7G students who had studied very little algebra previously were investigated first, as these give an indication of what beliefs about letters students bring with them from their previous experience (section 7.2). These included interpreting expressions like $2a$ as $2 + a$ rather than as a product ('sum'); working with coefficients, then putting a letter on the end of a number ('letters'); interpreting $2a$ as twenty-something, where the a stands for a digit, rather than a number ('code'); equating a with 1, b with 2, and so on ('alphabetic'); and substituting numbers inappropriately ('substitution').

The incidence of misconceptions before and after the graphic calculator modules among the whole student body was then discussed, together with a consideration of the effect of the graphic calculator teaching method on this incidence (section 7.3). It was found that the graphic calculator model and teaching method enabled many students to become aware of their misconceptions, and to change their constructs. Finally, misconceptions which occurred during the teaching process were investigated (section 7.4). These included 'product' errors, where expressions like $12pq$ were interpreted as $1 \times 2 \times p \times q$ or $21pq$, and interpreting letters as objects (both physical objects, such as $6a$ meaning six apples, and more abstract objects, such as $5x$ meaning five 'little x 's or $xxxxx$).

CHAPTER 8 CONCLUSIONS AND RECOMMENDATIONS

8.1 OVERVIEW

This research originated with my Year 9 student, Sally, who was so sure that $2x$ minus x equalled 2. During the course of this research, I found many students whose understanding of such expressions was similarly inadequate. These include the three Year 8 students, whose interpretations of letters I used to open this thesis: “I think that a and b are only letters that don’t mean anything.”, “ a and b are just fancy things at the end of a sum.” and “?” Contrast these remarks with that made by a student who had found meaning in what she was doing: “Oh! Wow! I never knew...!”

These examples all show why I felt this research was necessary. So many students are not able to find meaning in the symbols used in algebra, and hence struggle with algebra as a mathematical language. Yet if they cannot make sense of algebra, much of mathematics is likely to remain a closed book to them, and they will miss that sense of excitement the student quoted above experienced in making a new discovery. If the graphic calculator could help students to make their own discoveries and to understand more, then I felt this would be worth demonstrating.

The process by which this research came into being is described in detail in Chapter 1:

Introduction. My own classroom experience was the starting point, but a number of other strands contributed to my focus on my primary research question. Other strands were contributed by the work of Alan Graham and Mike Thomas on using the graphic calculator to enable students to learn algebra (1998; 2000a), the radical constructivists’ emphasis on

the need for students to construct their own learning (e.g. von Glasersfeld, 1991, 1995) and a paper by Margot Berger (1998). Berger led me to the work of Roy Pea (1985; 1987) in which he discussed the metaphors of amplification and cognitive change to describe how computers might help students to learn. Berger's paper also contained a quotation which was significant both in firing my imagination, and in leading me to the work of Lev Vygotsky:

If one changes the tools of thinking available to a child, his mind will have a radically different structure. (Berg, 1970: 164, cited in the *Afterword*, Vygotsky, 1978: 126)

Above all, the work of Lev Vygotsky provided the theoretical grounding for all the work that followed.

Eventually, these strands coalesced into this question:

- Is the graphic calculator a useful mediating tool for students in the early stages of forming a concept of a variable?

Sub-questions which then arose from this were:

- Does the model of a variable provided by the graphic calculator mediate successfully between students' initial interpretations of letters and an interpretation which will help their progress in algebra?
- If graphic calculator use proves helpful, what are the attributes of the graphic calculator which make it a suitable tool for students learning algebra?

In Chapter 2: *Review of the literature*, and in Chapter 4: *The graphic calculator: Mediating in a learning environment*, a theoretical model to explain how the graphic calculator might help students is discussed. This model is heavily dependent on certain of Vygotsky's theories:

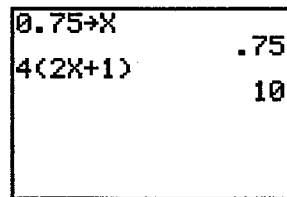
- Mediation of tools – the use of appropriate tools to carry out physical activities.
- Mediation of signs – the role of psychological ‘signs’ in enabling humans to develop higher mental functions, for instance, putting a knot in a handkerchief to aid the memory (Vygotsky’s example, 1978: 51) or putting a number into stores labelled with X and Y on a graphic calculator to give meaning to the symbols x and y in algebra.
- Concept formation – occurring through the interaction of everyday ideas (such as putting numbers into stores) and taught ‘scientific’ ideas (such as a variable).
- Key role of speech – the facilitation of such interaction by language and discussion.
- Zone of proximal development – the gap between what a student can do unaided and what s/he can do when supported.

Using Vygotsky's theory, I modelled the graphic calculator as a mediating physical tool.

Students start with the physical act of putting numbers into stores labelled with letters, then key in the operations to be carried out on that letter. When they press the ENTER key, the calculator returns a value. All this involves the student in using a physical tool, repeatedly carrying out the same actions, and talking or thinking about putting numbers into stores labelled with letters.

The student quoted below illustrates how the students in my case studies talked while using the graphic calculator. Claire and her partner were checking that $x = 0.75$ in the equation $4(2x + 1) = 10$, which they were trying to solve:

Figure 16: Calculator screen illustrating Claire's remarks



So let's try that. 0.75 STO, ALPHA X, ENTER, and then it's 4, open brackets, 2 ALPHA X, plus 1, close brackets, ENTER, is 10, so that's right. So x equals 0.75.
[Claire, Year 8 case study, taken from a transcript used in Chapter 5]

The graphic calculator is also a mediating 'sign'. A sign is a psychological or symbolic tool which enables us to extend our mental faculties in the way that a knot in a handkerchief or a list can help us to remember things. The most important signs are spoken language and writing. Algebra is also an example of a sign. The graphic calculator's screen, which displays numbers put into labelled stores, and the results of operating upon them, is another such sign. The screen display helps students to find out about algebraic operations, thus extending their mental faculties.

According to Vygotsky, true concept formation occurs when an everyday concept, such as a store containing a number, interacts with a scientific concept, such as a variable.

Concepts require both a root in everyday life and the kind of knowledge taught in school, and are formed by their interaction and fusion. The graphic calculator provides a locus for this meeting.

Language has a key function in this process. Vygotsky believed that concept formation could only occur through words, whether thought, spoken or written. The graphic calculator provides students with appropriate language to talk about algebraic expressions.

It enables them to articulate their ideas, so that they can examine them and then check with the calculator to see if they are correct.

The combination of two students working together, supported by a graphic calculator, is theorised as a zone of proximal development, in which the students are enabled to access ideas which they would otherwise have found beyond them. In Vygotsky's original theory, a student is supported by an adult or more able peer who enables the student to answer questions s/he could not have answered alone. In my theorisation, support is provided by a pair of students working together with a graphic calculator. This enables both students to reach a higher level of understanding than they would have done without the calculator or each other.

Other researchers provided a variety of metaphors which I used to describe how the graphic calculator could function as a tool for learning. Two of the most important of these for me in developing my model for how the graphic calculator functions as a tool for learning were those of amplification and cognitive change (Pea, 1985, 1987). A tool which amplifies allows us to do what we can do already, but to do it more quickly or more easily. I did not feel that the graphic calculator, used in the way described in this thesis, acted in that way. Indeed, students on occasion complained that they could have done their work more quickly without the calculator. A tool which allows cognitive change to occur, on the other hand, enables us to do something we were previously unable to do, and in so doing produces the opportunity for learning. This was demonstrated many times in the research conducted for this thesis, and is discussed in Chapter 5: *Evidence of cognitive change*.

Another important aspect considered in Chapters 2 and 4 is the process by which an artefact becomes a useful tool, known as ‘instrumentation’ (Artigue, 2002; Vérillon and Rabardel, 1995). Recent French research has emphasised that it is not enough to give students a new piece of technology or software, and then expect them to use it to help them with their mathematics. The unfamiliar technology or software will initially obscure the mathematics. It can take a considerable amount of time for ‘artefacts’ (technology/software before they have become useful tools) to become transparent, so that the mathematics can be revealed.

The way in which the graphic calculator was used in the classroom studies in this research minimises this process. The idea of a labelled store into which a number is put is easy to grasp, and operating on these stores is very straightforward. To illustrate, suppose the number 8 is to be put into the P store, to test whether $2P$ is the same as $P \times P$. The following screensnap demonstrates how this might be done:

Figure 17: Screensnap illustrating how students could gain an understanding of the expression $2P$

8→P	8
2P	16
P*P	64

The key strokes required to produce this screensnap are:

8 STORE ALPHA P ENTER
 2 ALPHA P ENTER
 ALPHA P * ALPHA P ENTER

Students found such exercises both straightforward and transparent: they could carry them out without difficulty and understood what they were doing. Operating the calculator did not obscure the mathematics for them, but revealed it.

Data of various types were collected during the two phases of classroom studies, as described in Chapter 3: *Research methodology and methods*. The first phase consisted of a series of case studies carried out at school G, a girls' grammar school. Initially a pilot study with a Year 9 class was conducted. This then led on to a much larger study with three classes of Year 7 students from the same school. One of these classes was followed up a year later, in the Year 8 case study. These three case studies provided much rich detail about what happened while students were working with the graphic calculators, particularly through the recordings made of students' discussions while working, interviews with them and with their teachers, and questionnaires to test their ability to answer standard algebra questions.

To see how well the findings of these case studies applied in schools more generally, a second phase of the research was conducted. This was a large-scale survey in four different schools from different areas of England, involving over 300 students.

Questionnaires were used in this survey, very similar to those used in the earlier case studies, to see how the students' understanding and skills changed after they used the graphic calculators and accompanying worksheets. Evidence of progress was found, both in terms of understanding and in skill in answering basic algebraic questions. This is particularly significant since these classes only spent about three hours on the graphic calculator work.

As a result of the classroom studies, I would suggest that the graphic calculator did indeed prove to be a useful mediating tool for students in their early encounters with algebra: that the model of a variable it provides enabled many of the students to form a meaningful understanding of a variable; and that it helped students to understand basic algebraic operations. These claims are supported by the findings related in Chapter 5: *Evidence of cognitive change* and Chapter 6: *Developments in students' understanding and skills*, and are reviewed in more detail in section 8.2 below. Through discussion and the physical activity of putting numbers into the calculator's stores, together with the feedback provided by the graphic calculator, the students came to a fuller understanding of what letters mean and how they are used in algebra, as the transcripts in Chapter 5 and the statistical evidence given in Chapter 6 show.

I also found evidence in my data of areas of misconception in these students, which is reported in Chapter 7: *Misconceptions*. Students' lack of comprehension of letters and algebraic expressions was one of the starting points for this research, but the wide range of misconceptions shown and their persistence still surprised me. The misconceptions I found were not particular to any one class or school, and are therefore likely to be widespread in children of this age. It would appear that the majority of students in the 10-14 age group hold some misconceptions, despite previous exposure to algebra.

8.2 RESULTS OF THE CLASSROOM STUDIES

In Chapter 3: *Research methodology and methods*, the classroom studies carried out during this research are described. The methodology was of a mixed type, with qualitative data collection in the case studies used to define themes and categories, which were then made the basis of a quantitative survey. The case studies enabled graphic calculator mediation to

be studied at the level of the interaction within individual pairs of students as they worked together with the graphic calculator on the teaching materials. This was also investigated through questionnaires and interviews with both students and teachers. The themes which emerged from these case studies as significant were cognitive change in the students, their progress in understanding and skill, and the misconceptions they harboured. These themes are the subjects of the three main findings chapters, 5, 6 and 7. The survey data were used to explore students' progress and their misconceptions over a wider sample of students, to see if the results of the case studies could be more widely generalised.

Cognitive change occurring or failing to occur in the students' thinking during the graphic calculator teaching modules was the subject of Chapter 5: *Evidence of cognitive change*.

Cognitive change was defined as occurring if there was evidence that a student's thinking had been restructured in some way (Pea, 1985, 1987). The recordings, made during the Year 7, 8 and 9 case studies, of pairs of students working together with one graphic calculator between them, were used to provide this evidence. Additional evidence was also obtained from interviews held with students and teachers, and the questionnaires students answered before and after the teaching modules. Examples of the occurrence of cognitive change, and occasions where it failed to occur, are reported. An example of cognitive change which did occur, given in section 5.2.2, is the account of Sofia and Chantelle's, and Megan and Lucy's, discussions on whether $2P = P \times P$ or $P + P$.

Two requirements were noted for cognitive change to occur. The first was that the problem was within the zone of proximal development defined by the two students and the graphic calculator: that is, whether the students *could* answer the problem when supported by the graphic calculator. Some of the examples given in Chapter 5 show students

working on problems which were simply beyond what they could access even with support, as in section 5.2.5, where Fran and Rebecca are seen struggling with equations they do not really understand. The second requirement was that the students used the graphic calculator in a way that would enable it to mediate between them and the algebra. This means that they did, in fact, put numbers into the appropriate stores and evaluate the expressions. Occasions were described where cognitive change failed to occur because the students simply used the calculator to re-evaluate the expression incorrectly, without using the appropriate stores. This is seen in section 5.2.4, where Claire and Briony evaluated incorrectly an expression containing brackets both in discussion, and when using the graphic calculator.

The results reported in this chapter support the claim that the graphic calculator does enable cognitive change to occur, and that cognitive change is a useful way of describing what happens when pairs of students work together with the calculator. Amplification is not an appropriate metaphor, because students were not enabled to do things they could do already more quickly or easily. In fact, the reverse of amplification is seen in some of the transcripts in Chapter 5, where students can be observed struggling with problems for long periods before resolution occurred, or they gave up and moved onto something else.

Analyses of the data which support the claim that the students did indeed make progress in understanding and skill level are reported in Chapter 6: *Developments in students' understanding and skills*. Progress in understanding was first tested using a direct question asking students what they understood the letters in algebraic expressions to mean. After the graphic calculator teaching modules, a smaller proportion of the students interpreted

letters in a non-numerical way, and a greater proportion interpreted them in an algebraic way.

Progress in understanding was also tested using a framework devised by Küchemann (1981). This framework gives six different categories of interpretation of a letter which Küchemann and his colleagues observed in a survey of over 3000 students aged 13 to 15 years in the late 1970s. Three of these categories are ways to avoid working with letters at all, and were classed by Küchemann as level 1. Students showing level 2 understanding interpret letters in these ways also, but can work with expressions which are structurally more difficult. When they reach level 3, students make a breakthrough in that they interpret letters as specific unknown numbers. By level 4, students are beginning to understand variables as generalised numbers, or even as true variables, and can work with expressions showing greater structural complexity. A majority of the students who participated in the classroom studies showed a rise in their level on Küchemann's hierarchy, particularly those who started at level 1 or below. The increase these particular students showed was statistically highly significant.

Students' verbal interpretation of letters, as given by their response to a direct question about the meaning of letters, was then compared with the level they reached on Küchemann's framework. Most students showed consistent interpretations and levels. However, some did not, and these cases were discussed further.

Students' ability to handle algebraic expressions and equations in a proceptual way was also considered. Understanding an expression proceptually means seeing it as an object which can be operated on, as well as a collection of procedures (Tall and Thomas, 1991),

which is necessary if students are to succeed in understanding algebra beyond the most basic level. The proportion of students showing proceptual understanding corresponded to their level on Küchemann's framework, with those on the lowest levels lacking proceptual understanding while most of the students on the higher levels showed such understanding.

Students' general ability to answer algebraic questions was also considered. It was found that most students made at least some progress, and some, particularly the youngest students and/or those who started with a low level of understanding made exceptional progress. Some Year 8 students did not make much progress, or even dropped back in their results, however. Given the nature of the data collected, it was not possible to be sure why this should have occurred, but it is possible that factors outside the teaching modules considered here could be significant. It is also possible that this is not a real effect at all, but was caused by students who already had a good understanding not having the questions available to them which would have shown any improvements made.

Finally, students' misconceptions were considered in Chapter 7: *Misconceptions*, together with the role of the graphic calculator in remediating these. Misconceptions were found to be widespread and very persistent - far more so than is perhaps realised by teachers.

Misconceptions were divided into those which children brought with them as a result of their previous experience, and those which appeared during the teaching modules. Major types of misconceptions were held by surprisingly large proportions of the students, and were shown by students with the greatest lengths of previous exposure to algebra as much as by those who had not studied algebra previously. Some of these misconceptions were susceptible to the model of a variable provided by the graphic calculator, such as equating $2a$ with $2 + a$ and interpreting $6a$ as sixty-something, while others proved less tractable.

The two types of misconceptions which appeared during the teaching modules involved interpreting letters as objects, and misunderstanding products involving coefficients with two digits. Some students interpreted expressions like $5x$ as five little x 's, a form of 'objectifying' x which worked perfectly well when the coefficient was an integer, but not otherwise. Writing $xxxxx$ for $5x$ makes sense (even if technically it is incorrect); but, as Rebecca and Fran discovered (section 7.4.2), this does not help in dealing with $1.5x$. Misunderstanding products occurred in two different ways: interpreting $24pq$ as either $2 \times 4 \times p \times q$ or as $42pq$. Such errors were not apparent prior to the teaching modules, and appeared, I would suggest, as a result of students misunderstanding what their teachers said to them, or applying rules inappropriately.

8.3 IMPLICATIONS FOR THE CLASSROOM

The findings of this research have several implications for classroom work. The results indicate that using graphics calculators as a way of introducing children to algebra is worth pursuing. The graphic calculator model of a variable has the potential to help students begin to form a concept for a variable, and to understand basic algebraic conventions. Graphic calculators are easily obtainable, and are straightforward for the students to operate. The model of a variable as a number put into a store, labelled with a letter, is easy to understand and work with. The model emphasises that the letters used represent numbers, that those numbers can change, and that the letters used are arbitrary. This is empowering for students, many of whom believe the letters to be empty symbols, and algebraic expressions to lack any real meaning. Furthermore, most of the students in this research found the graphic calculator method of working helpful and enjoyed it, as discussed in section 4.4.4.

Most of the studies described in this thesis did not last very long, with the majority of classes only spending three or four hours on this work. If the graphic calculator method was used as the basis for all the work students did in algebra, then this would give them opportunities over time to develop a sound concept of a variable, and to work out how to perform basic operations on variables. The Year 8G group, which was part of the main case study in Phase I of this research, used graphic calculators for all their work in algebra throughout Year 7, and into Year 8. Their understanding had developed noticeably during that time, as is exemplified in section 5.2.4. One pair of students, Claire and Briony, as Year 7 students failed to use brackets correctly, but as Year 8 students showed a much improved understanding of the function of brackets. By the Year 8 case study, the class as a whole was able to answer questions of greatly increased difficulty compared to a year earlier.

The examples of cognitive change in Chapter 5, and the statistics about the improvements in understanding and skill in Chapter 6, all illustrate that graphic calculator use does help students to learn basic algebra. Copying screensnaps helps students both to internalise that letters represent numbers and to clarify what algebraic expressions mean. Having a means of checking their work in this way also means that students are less likely to let misunderstandings pass unchecked. However, as was apparent from the classroom transcripts, it is important for teachers to be aware of the need for students to use the calculators appropriately: that is, by actually putting numbers into the stores, not just using them to evaluate the answers to operations, which may mean that previous errors are repeated. It is also necessary for the teacher to bring out important points and to expose misconceptions with groups of students and with the whole class. The discussion which

this generates is part of enabling students to articulate their ideas, and hence to develop a deeper understanding.

In order to know which points need further elaboration, and which misconceptions need general discussion, teachers need to be aware of the depth of ignorance shown by many students, and the number and variety of misconceptions they hold. Of course, teachers usually realise that their students have problems, but it is possible for the real extent of ignorance to be hidden. Many students do not answer questions in class if they can avoid it. Written work may be marked and then discussed in class, but again, students tend to hide their lack of understanding. In support of this contention, *only 18% of the students* who participated in the case studies and the survey described in this thesis *showed no misconceptions at all*, and this includes the older students as much as the younger ones. To restate this: *four out of five of all the students held misconceptions about the nature of letters or algebraic expressions*. Even among the Year 9 students, who took part in the pilot study, the proportion of students showing at least one misconception *was greater than 50%*.

Teachers also need to be aware of the potential for new misunderstandings to arise. The 'product' misconception in particular, appeared to originate from teachers saying that when two algebraic symbols are written next to each other without an operation between them, this means they are to be multiplied. Some students interpreted this to mean that 24 can mean 2×4 in algebra. Although they would reject this as nonsensical in an arithmetic context, at this stage their understanding is not yet secure enough for them to reject such ideas when letters are involved as well. Challenging ideas like this with the graphic calculator enables students to see that this is an inappropriate extension of a rule.

8.4 STRENGTHS AND LIMITATIONS OF THIS STUDY, RECOMMENDATIONS FOR FUTURE RESEARCH

This research focused on students' early encounters with algebra, and it has already been reported in the following papers, seminars and conference proceedings: (Gage, 1999a, b, 2001, 2002a, b, 2003a, b).

The graphic calculator model is very suitable for helping students to find meaning in their first encounters with algebraic letters, as it emphasises that letters represent numbers, which are arbitrary and which can change. The model is difficult to extend to the stage of a true variable, however, where the letter does not represent any particular value, but can represent any value in the domain of a function. A calculator store has to contain a specific number at any one time, even if that number is random and perhaps unknown to the student, as it would be if produced by a random number generator. All models possess limitations, and eventually a point comes when these have to be made explicit to the students, so that they can move beyond the model, developing a deeper, more abstract concept. However, I do not see this as a limitation on this particular piece of research, because it was explicitly focused on students' early understanding of a variable.

In the classroom studies, data were gathered from over 400 students over a period of three years. The mix of qualitative and quantitative data collected from these students allowed themes to emerge, which could then be further tested with a wider sample of students. The initial phase of case studies provided rich detail, partly because of the qualitative nature of the data collection at this stage, and partly because it took place at the school in which I worked at the time. I knew what was happening in the various classrooms and could talk

to the students and teachers while they worked with the graphic calculators. Emerging findings could be tested immediately with the participants, and verified or dismissed.

It may be, however, that this also led to 'researcher effects'; that is, the results were affected by my presence. The other teachers involved were both aware that this was a research project whose results were important to me. Students would undoubtedly have been affected in some way by the data collection process, realising that these lessons were not in the ordinary run of things. The attitudes of both students and teachers may have been influenced by these factors.

In any case, data from one school, particularly a school which was both selective and single-sex, would not have been adequate for establishing wider generalisations. The second phase of the research, the survey in the four non-selective, mixed schools therefore added much needed generality. In addition to extending the generalisability of my findings, I did not visit these schools at all, so that the 'researcher effect' would have been much less. Students would have been less affected by the data collection process, since they just answered questionnaires, and it is doubtful that they would have identified it as a project in which their own teacher had a vested interest.

There were, however, problems caused by my not being able to visit these schools. I could not be sure of the extent to which the instructions I sent out with the classroom materials were observed. I cannot be certain, for instance, that students worked in pairs with a single graphic calculator between them, nor do I know if the questionnaires the students answered were really the work of individuals, or if students collaborated on their answers or teachers helped them, despite my instructions to the contrary.

The data from these schools would have been greatly strengthened if I could have audio-taped conversations between pairs of students while they worked, as I did in school *G*. Interviews with the students would also have contributed to my understanding of their thought processes. In particular, it would have been very useful to listen to the discussions between pairs of Year 8 students, and to talk to them after they finished the graphic calculator work. This would have enabled me to see why some of them appeared not to make progress, or even to drop back in their understanding and/or skills.

The relative contributions of the schools in the second phase is unbalanced, since most of the students were from the middle school, school *B*. School *A* contributed two classes, one of which was very small, and schools *C* and *D* contributed one class each. A more balanced study would have had a more equal number of classes from each school, which also showed a balance across the age range. This particular limitation occurred because schools chose to participate, rather than my being able to select a sample.

It would be good to see another survey, in a well-chosen sample of schools, which repeats the research carried out in Phase II. As well as a questionnaire survey, such research could include videotaped classroom observations, audiotapes of students working together with the graphic calculator, and interviews with teachers and students afterwards. In follow-up interviews, students could then be asked what they meant in their spoken remarks and written statements, where these posed problems in interpretation. They could also be asked what they felt had gone wrong, if they failed to make progress.

There are several specific issues mentioned elsewhere in this thesis which could benefit from further research. These include the connection between a student's level on

Küchemann's framework, and the degree to which they made progress in their understanding and skill measured in other ways. Inconsistencies between a student's levels on different analyses is also an area which could be further clarified. Both these issues would need separate sets of questions to test the various measures and levels, to see if agreements and inconsistencies were an artefact of the research method used or not.

In addition, further research could be done on students' misconceptions to give us greater understanding of what they understand and what they do not understand. It would be useful to know where an error is the result of a prior misconception; where it comes from a rule being inappropriately extended; where it is the result of insufficient understanding of the question; where it is a case of a student being extended beyond their zone of proximal development (that is, beyond the level at which they can cope, even with help); or where it is inadvertently introduced by the teaching method.

8.5 FINAL SUMMARY

Finally, I would like to return to the questions used to focus this research:

- Is the graphic calculator a useful mediating tool for students in the early stages of forming a concept of a variable?
- Does the model of a variable provided by a graphic calculator mediate successfully between students' initial interpretations of letters and an interpretation which will help their progress in algebra?
- If graphic calculator use proves helpful, what are the attributes of the graphic calculator which make it a suitable tool for students learning algebra?

The findings of this research show that this method is effective in helping students to gain a greater understanding of a variable, and to work more effectively with algebraic expressions. Most students showed improvements in their understanding and skill, and those whose understanding was initially small showed exceptional progress. I would therefore argue that graphic calculators do provide useful, mediating tools for students in the early stages of forming a concept of a variable.

The model of a variable the graphic calculator provides is transparent and easily understood, and, as has been seen in Chapters 5, 6 and 7, does have the power to mediate successfully between students' initial interpretations of letters and the kind of interpretation they will need in order to become fluent users of algebra. It is a model which is straightforward to incorporate into classroom materials, or to use with existing text book exercises. Little training is required for students to put numbers into the calculator stores, and then to use these to discover the properties of algebraic expressions. The examples in Chapter 5 show the students grappling with the problems caused by the algebra, not with problems caused by the graphic calculator.

The attributes of the graphic calculator which make it a suitable tool for students learning algebra are various. A graphic calculator is a relatively inexpensive, small, personal instrument, affording students a domain for testing conjectures and ideas in a private, safe environment. Like the computer, this domain is provisional: it can be quickly changed and is not permanent. Once something is written down, there is a tendency to think it is then permanent: students write down their answers to a problem, and move on to the next question. Keeping an answer provisional is a way of ensuring that students do not move on too soon. Examples of students staying with a problem and eventually solving it are

described in Chapter 5: *Evidence of cognitive change*. The graphic calculator, by providing a locus for cognitive change, enables students to learn.

In this thesis, a model to explain how this learning might occur has been described. This model is dependent on Vygotsky's work, particularly on the mediation of tools and signs. The calculator is theorised as both a mediating physical tool, and a mediating psychological sign. In both these capacities, it gives students the opportunity to extend their combined zone of proximal development so that, working in pairs with the calculator, they are able to solve problems which neither would have solved alone or without the calculator. In Vygotsky's original theory, help and support are provided by an adult or more proficient peer; in my theorisation, help and support are provided by the student pair (neither of whom needs to be more proficient) and by the graphic calculator. The calculator provides a physical model of a variable, physical acts to get them started on a question, language to enable the students to discuss the problem at hand, and feedback on whether what they are doing is correct or not.

It has been my intention in this thesis to demonstrate that the graphic calculator *is* a useful mediating tool for students in the early stages of forming a concept of a variable, and that this enables them to make progress in their understanding of and skill with algebra. Students' learning is intimately bound up with the actions they carry out with the graphic calculator, and the words with which they articulate their ideas. The graphic calculator way of working enables students to talk with understanding about what letters and expressions mean, thus facilitating cognitive change.

I would like to close with words from Vygotsky, whose work has made so many of the arguments in this thesis possible. Initially, development is caused by action: we developed from apes through the use of tools. However, development is continued by the word: through language (spoken and written) and thought which become part of the continuing action – the word and the act are in a dialectic relationship in which neither supersedes the other, but together they form a synthesis greater than either. Without tools, we can do nothing. Without words, we cannot conceptualise what we do with our tools. With both:

... since we wanted to express all this in one short formula, in one sentence, we might put it thus: if *at the beginning* of development there stands the act, independent of the word, then at the end of it there stands the word which becomes the act, the word which makes man's action free. (*Tool and Symbol in Child Development*, by Lev Vygotsky and Alexander Luria in van der Veer and Valsiner, 1994: 170, original italics)

ANNEXES

ANNEX I: CLASSROOM MATERIALS AND TEACHERS' NOTES

The materials and notes produced for the Year 6-8 survey are reproduced here, as these were typical of the earlier case studies, and were put into a complete form to be sent out to unknown teachers.

General instructions for Year 6-8 Algebra Project (2001/02)

What to do, when to do it, and how to do it.

If anything at all is unclear, or you want to ask questions about anything, please e-mail me on jag55@cam.ac.uk or jag43@tutor.open.ac.uk, or phone me on 01353 666426 or 0777 189 1776. I am most anxious that this shouldn't cause you unnecessary extra work.

Included in these instructions are three questionnaires for the students to complete and a questionnaire for any teacher involved. There are also teacher notes and worksheets for the actual lessons.

Initial student questionnaire

- Do this *first*, immediately before starting work on the topic – it will provide me with a baseline against which to measure subsequent results for your students. The quality of their answers doesn't matter – I just need to know where they are before they start this work.
- Tell the students that they are doing the questionnaires because their work is part of a research project into how Year 6, 7 and 8 students learn algebra, done by the Open University, and that their views are very important.

- ❑ Please make sure students do it individually, without talking to each other about their answers – it is important to me that I get each student's own answers rather than a joint effort.
- ❑ They should not use a calculator of any kind for this.
- ❑ Please don't explain any of the algebra questions to the students – if they ask how to do them, just say I want to know what they think, and they should put whatever they think.
- ❑ Give the students as long as they need to complete the questionnaires – they can give their questionnaires in once they have finished, but should have something else to get on with so they don't disturb others still working on it.

Working through the topic

- ❑ Use the teachers' notes provided to prepare the topic.
- ❑ Please encourage students to work in pairs, so that they can discuss their work as they do it.
The discussion is an important part of the learning approach.
- ❑ All worksheets provided for class work.
- ❑ Don't worry if you don't get through everything.
- ❑ If you have a student who does need extra work, use anything you like from your normal scheme of work.
- ❑ If you want to set homework on this topic, use anything you like from your normal scheme of work.
- ❑ Please **DON'T** teach with an eye on the questionnaires. There are questions in them which are deliberately not taught on the worksheets, to see how students' concept development helps them in gaining algebraic skills. It would be best if you ignored the questionnaires completely, apart from actually administering them, and then sending them off to me. If you want to look at them in more detail, please do it after you've finished the topic.
- ❑ You might want to complete the teacher's questionnaire as you work through the topic, however. I just need full, accurate answers to this.

Second questionnaire

- ☐ Same conditions as first one.
- ☐ This should be given to the students to do as soon as possible at the end of the topic.

Teacher's questionnaire

- ☐ Please get all teachers involved to complete this as soon as possible after the end of the topic.
- ☐ Same conditions as for the students – don't confer, take as much time as it needs.
- ☐ Please tell me exactly what you think. I won't be offended if you want to be critical!

Third questionnaire

- ☐ To be completed about 6 weeks after the end of the topic.
- ☐ Same conditions as before.

Return of materials

- ☐ Send all completed questionnaires to me, preferably as you complete them, so I can be getting on with data analysis, to:

Jenny Gage
MMP
Centre for Mathematical Sciences
University of Cambridge
Wilberforce Road
Cambridge
CB3 0WA

Finally, many thanks for agreeing to use my materials, and to take part in my research. I am extremely grateful to you and to your students. I shall be writing this work up for a teachers' journal, and I will send you a copy of this once it has been accepted. I will also be writing it up for an academic journal or conference, and will send you a copy of this if you tell me you would be interested. This will also, of course, form an important part of my PhD thesis.

Teachers' Notes

Resources needed: Class set of graphic calculators (one between two is enough)

Worksheets provided

Objectives: Discover that numbers can be stored in letter stores on the GC

Discover how to produce expressions

Discover how to evaluate simple expressions

Find equivalent expressions

First double lesson

Initial activity (whole class)

The purpose of this activity is to familiarise the class with the calculators, and introduce them to the idea of storing numbers in the calculator's labelled stores. A useful way of visualising them is as 'boxes' with labels on them into which numbers can be put.

- If this is the class's first introduction to the graphic calculator, go through main key locations (this will vary according to which model you are using, but the notes here will give you an idea of what I mean by this):
 - grey numeric keys
 - blue operation keys and ENTER key (tells the calculator to carry out your instructions)
 - black keys giving additional operations (like squaring and brackets), new operations (like sin and cos) and keys which enable you to access many menus on the calculator
 - blue cursor (arrow) keys to move about the screen
 - yellow 2nd key to access items in yellow above keys

- black ALPHA key to access white letters on right above keys
- Explain that the calculator has memories or stores labelled *A*, *B*, *C*, ... which can have numbers put into them.
- To store eg. 3 in *A*, use this key sequence (substitute the appropriate key sequence for the model of calculator you are using):

ON

CLEAR

3 STO →

ALPHA A

Now key in ALPHA, A, ENTER. Your screen should look like this:

3→A	3
A	3

(Show on board – make sure everyone has managed to do this correctly, and make the point that CLEAR removes the display).

Class practice

- Store numbers of your choice in *A*, *B* and *C*. CLEAR the screen and check that when you press ALPHA, A, ENTER etc you get your numbers back. For example, store 3 in *A*, 23 in *B* and 7 in *C*, CLEAR the screen and see if you can produce the screen below:

A	6
B	23
C	7

- Now get the class to enter values as in the next screensnap:

A	3
B	4
C	2

and CLEAR their screens again. Now press 2, A, ENTER. What happens? Find out what happens if you enter 3B, 5C, etc. What is the calculator doing?

- Get the class to *predict* values for $A + 1$, $B - 2$, $C/3$, etc. They should then use the calculators to *check* their predictions. Do quite a lot of examples here, keeping them very easy – this predict and check sequence is important, and whole class activity will make sure everyone knows what they are doing before they start work on the worksheets.
- The class can then work on the *Screensnaps* worksheet, followed by *Screensnap Questions*, and *Harder Screensnaps*, as you feel appropriate.
- Discussion issues to raise at suitable points in the sequence of lessons (and you will probably find others arising during the lessons, or may want to add other issues of your own):
 - how many different stores does the calculator have?
 - what does the calculator do if you key 3A into it? 4C? etc
 - what does AB mean to the calculator? and BA?
 - if $AB = 20$, say, how many different ways can you produce this?
 - do you have to put whole numbers into the calculator stores?
 - can the same letter be used for different numbers at the same time?
 - can different letters be used for the same number at the same time?

Lesson 1 worksheets**Screensnaps**

On this worksheet are a number of calculator screens (“screensnaps”). You have to put the right numbers into the lettered stores so you can copy each screen on your own calculator.

Write down the number(s) you put into the store(s) in each case.

1.

P-2	4
-----	---

2.

12+J	63
------	----

3.

30-E	6
------	---

4.

66	30
----	----

5.

F10	250
-----	-----

6.

D/2	1
-----	---

7.

M/9	4
-----	---

8.

12/P	4
------	---

9.

BA	8
----	---

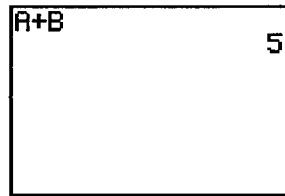
10.

AB	8
----	---

Screensnap Questions

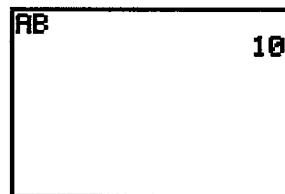
Make the screensnap first in each case, then answer the questions.

A



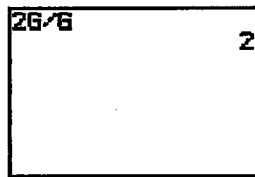
- 1 Can you make this screensnap in three different ways? Write down the numbers you use.
- 2 Can you make this screensnap without using whole numbers in the A and B stores? Write down the numbers you use.
- 3 Draw the screensnap for $B + A$.

B



- 1 Can you make this screensnap in three different ways? Write down the numbers you use.
- 2 Can you make this screensnap without using whole numbers in A and B ?
- 3 Draw the screensnap for BA .

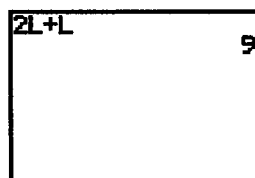
C



Make this screensnap in three different ways. Write down the numbers you use.

What would the screensnap for G/G look like? Draw it.

D



Try to make a different screensnap which only uses the letter L .

It should have the same number in the L store as this one, and also give 9 when you press the ENTER key. Draw it.

Harder Screensnaps

On this worksheet are a number of calculator screens (“screensnaps”). You have to put the right numbers into the lettered stores so you can copy each screen on your own calculator.

Write down the number(s) you put into the store(s) in each case.

$7+2R$	19
--------	----

$9+5D$	39
--------	----

$10B-13$	67
----------	----

$100N-100$	1800
------------	------

$1000Z-1000$	37000
--------------	-------

$10Q-10$	15
----------	----

$30-2X$	10
---------	----

$300-2T$	180
----------	-----

$12A$	-24
-------	-----

$3X+1$	-29
--------	-----

$20E/10$	21
----------	----

$2F/3-1$	3
----------	---

$3A/7-2$	1
----------	---

$3*(D+4)$	27
-----------	----

$5(2B-1)$	35
-----------	----

Second and third double lessons

Worksheets available: *Screensnaps and Stars*

Investigating More Complicated Expressions

Stories and Formulas

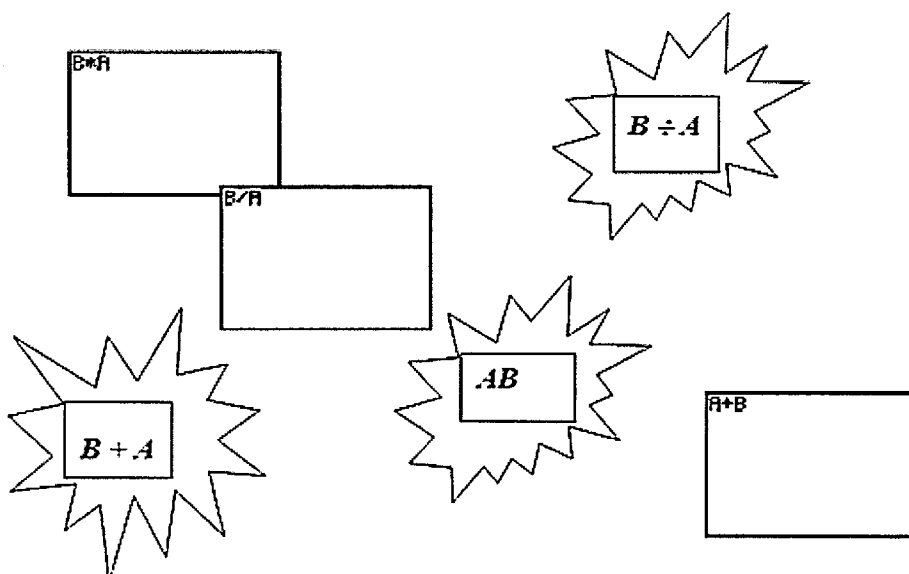
From this point on, use the worksheets provided as you feel is appropriate for your class –

I have deliberately produced more material than you are likely to want so that you can choose which worksheets you want to use. Of course, you may well want to use worksheets or exercises of your own instead of some of these. All I ask is that you make sure the students use the graphic calculators to check all their work as they do it.

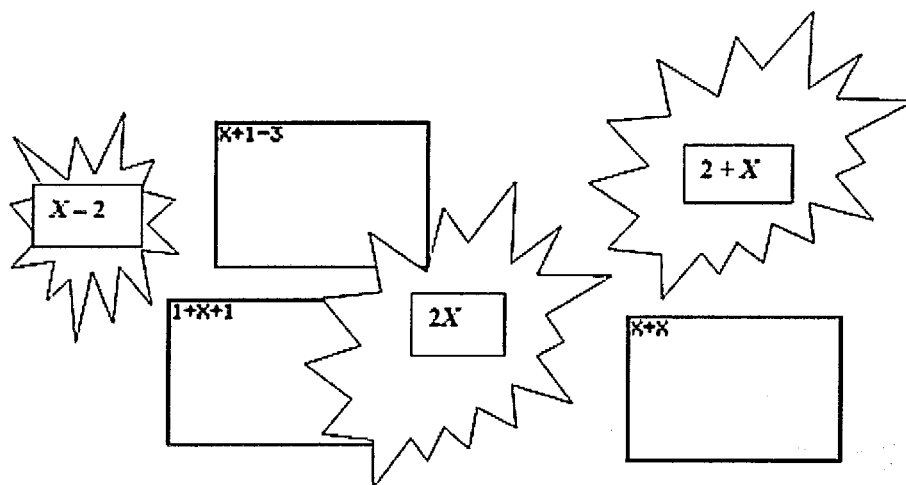
Students should work in pairs as far as possible, but there may be times when you want them to work as a whole class to discuss points arising, or to introduce a new idea.

Lesson 2 and 3 worksheets**Screensnaps and Stars**

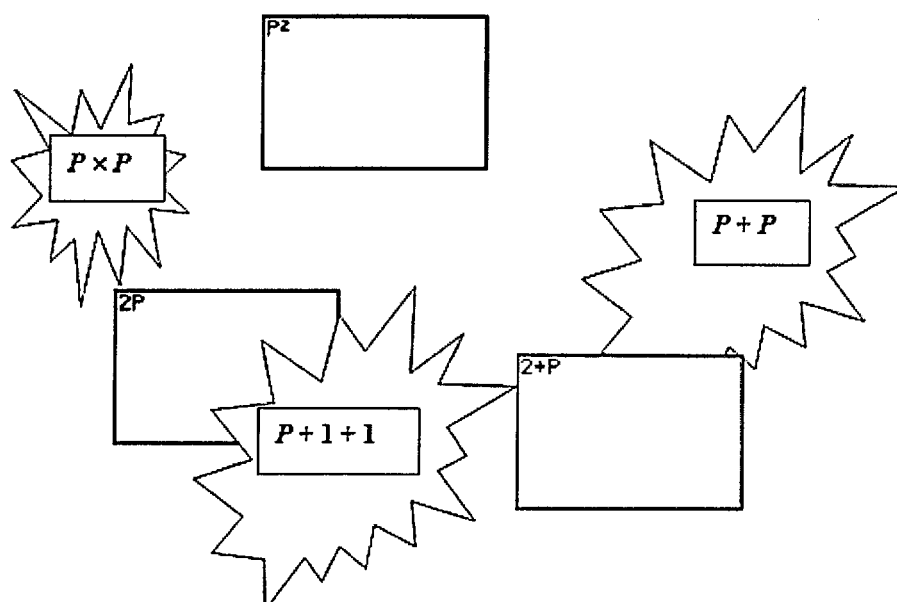
- A Look at these three screensnaps (* means multiply, the same as on a computer) and three stars. Pair up screensnaps and stars which show the same expression. Use the graphic calculator to check that you've got your pairs right (it doesn't matter what numbers you use for B and A , but you should avoid 1 and 2). You should get the same value when you press ENTER if your expressions are the same.



- B Do the same with these, using the graphic calculator to check that your pairs match.



- C Do the same with these, using the graphic calculator to check that your pairs match.



1. The first part of the report is a summary of the work done during the year.

2. The second part is a list of the work done during the year.

3. The third part is a list of the work done during the year.

4. The fourth part is a list of the work done during the year.

5. The fifth part is a list of the work done during the year.

6. The sixth part is a list of the work done during the year.

7. The seventh part is a list of the work done during the year.

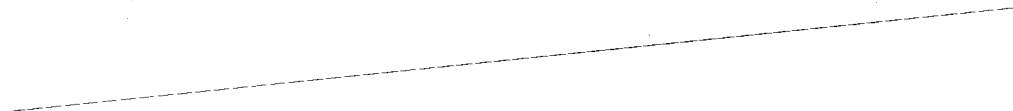
8. The eighth part is a list of the work done during the year.

9. The ninth part is a list of the work done during the year.

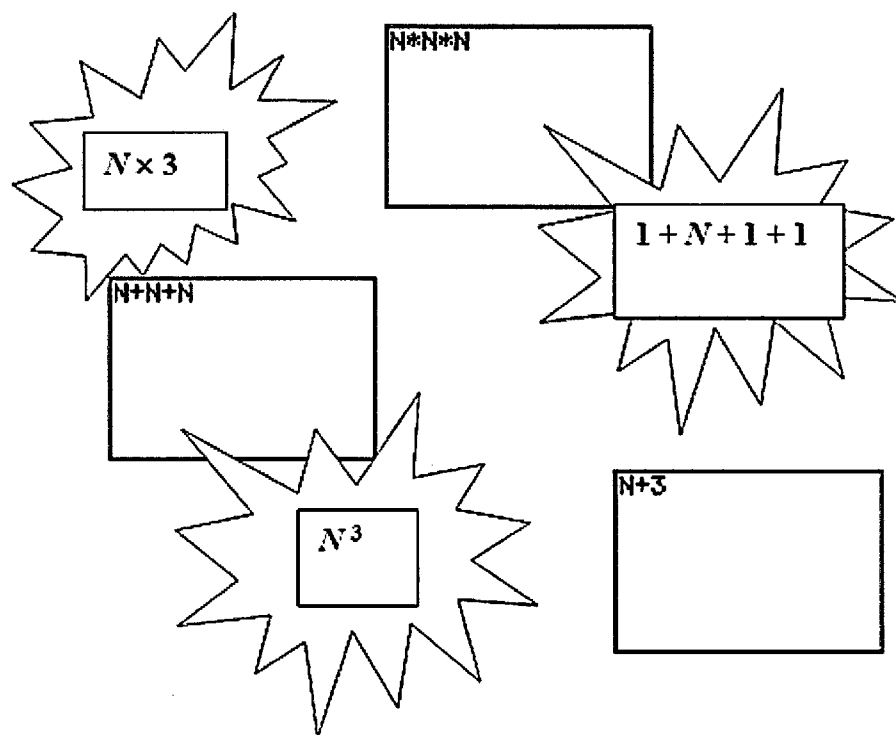
10. The tenth part is a list of the work done during the year.

11. The eleventh part is a list of the work done during the year.

12. The twelfth part is a list of the work done during the year.

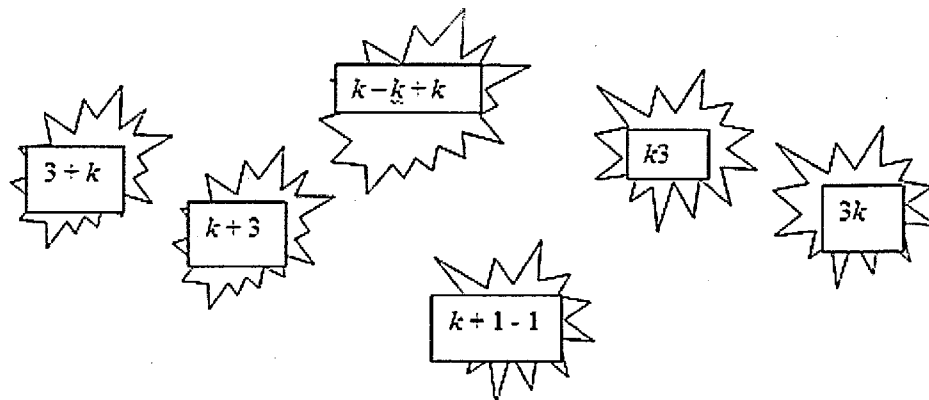


- D Do the same with these screensnaps and stars, using the graphic calculator to check your pairs are the same.

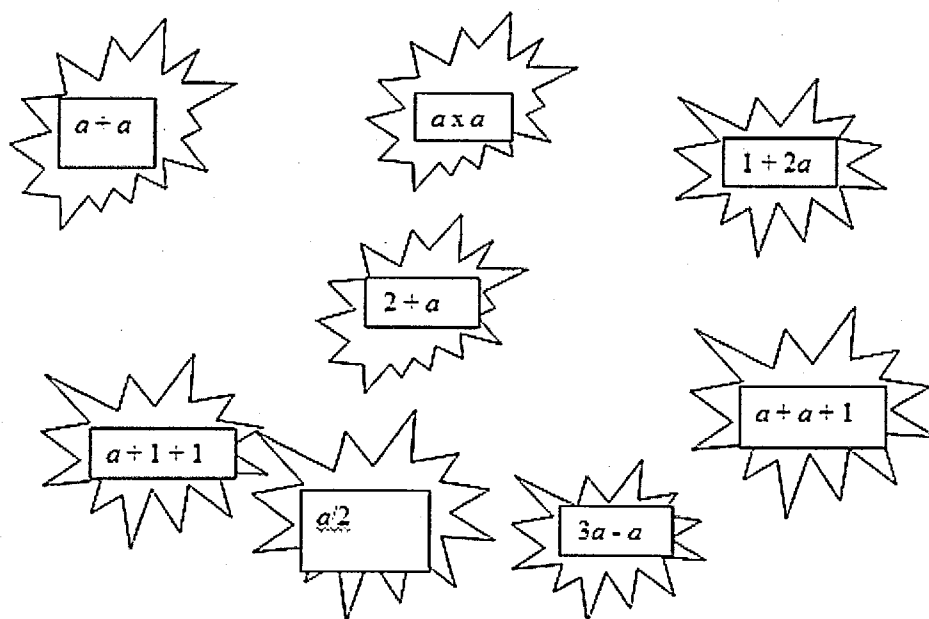


Investigating More Complicated Expressions

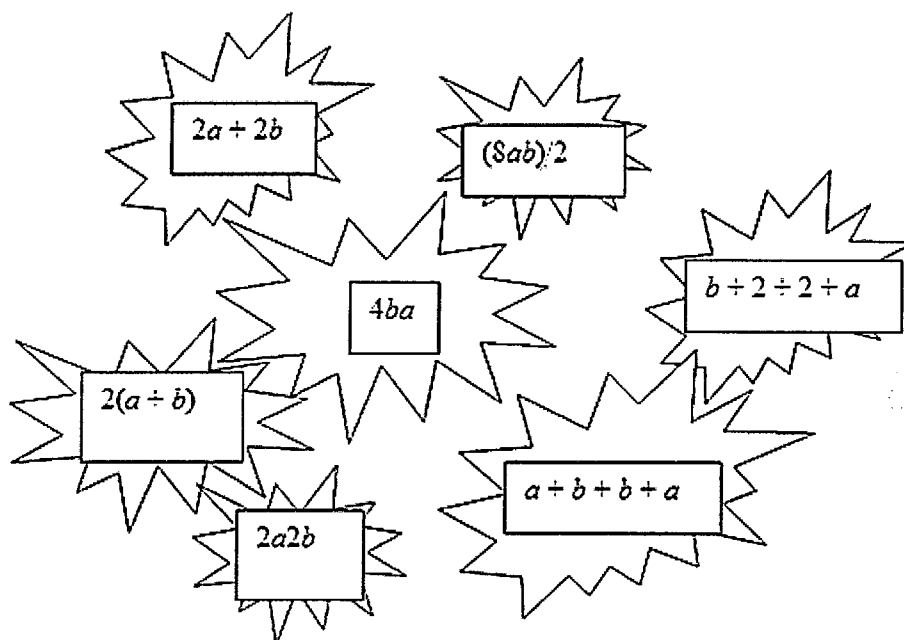
- A** There are three pairs of expressions which are always equal in these stars. Match them up. Use the graphic calculator to check your work.



- B** Look at this group of expressions. Match them into pairs. There are two odd ones out - which are they? Use the graphic calculator to check your work.



- C** Here is another group of expressions. Here there are two sets of three expressions which are always equal to each other, and one odd one out. Find the two sets, and the odd one out. Use the graphic calculator to check your work.



- D** Match expressions from the left-hand list with expressions from the right-hand list.

Which ones don't have an exact match? Can you write any of these expressions more simply? Use the graphic calculator to check your work.

$$1 + p + 1 - 1 + 1$$

$$1 - p + 1$$

$$p - 1 + p$$

$$p + p$$

$$p - 1 - 1$$

$$p - 2$$

$$2p$$

$$2 + p$$

$$2 - p$$

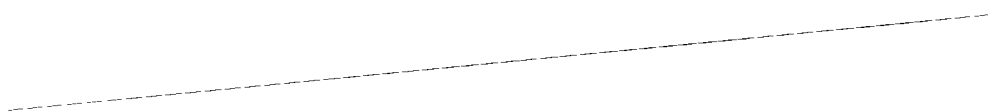
$$p \times p$$

(

(

(

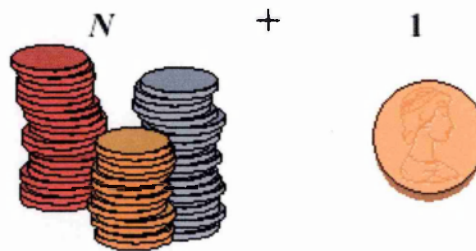
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Stories and Formulas

A “Sam adds a coin to a large pile of coins.”

We don’t have to know how many coins there are in the pile – we can just represent them with N , to signify a number of coins.



1. Draw the screensnap to represent this.
2. Write down the formula you get for the total number of coins, T , if you add 1 coin to the pile.
3. If $N = 200$, what is T , the total number of coins?
4. If $T = 300$, what is N ?

B “Emma has a large collection of CDs which she keeps in a rack, but two are lying on the floor, separate from the rest.”

Use N to represent the number of CDs in the *whole collection*.

1. Draw the rack of CDs with the two on the floor shown separately.
2. Draw a screensnap to represent the number of CDs in the rack.
3. Write what is on your screensnap as a formula, where T represents the number of CDs in the rack.

4. If $N = 128$, what is T ?

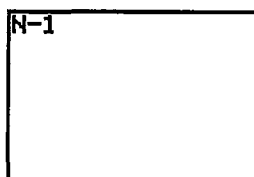
5. If $T = 135$, what is N ?

C “Tim and Nisha have equal numbers of sweets, which they put into one bag.”

Use N to represent the number of sweets each has to start with.

1. Draw a picture of this.
2. Draw a screensnap to represent it.
3. Write what is on your screensnap as a formula for the total number of sweets in the bag, T .
4. If $N = 25$, what is T ?
5. If $T = 76$, what is N ?
6. Draw a picture to show the bag of sweets if three more sweets are now added.
7. Draw a screensnap for this.
8. Write down the new formula for the total number of sweets in the bag, making sure that your formula is as simple as you can make it.
9. If $N = 12$, what is T now?
10. If $T = 48$, what is N ?

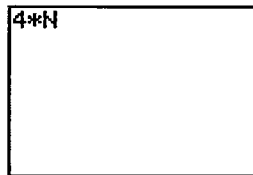
D



1. Write a story of your own which could be represented by this screensnap, making it clear what number N represents and what the total T represents.
2. Draw a picture of your story.

3. Write it as a formula for T .
4. Work out T if $N = 25$.
5. Work out N if $T = 38$.

E



1. Write a story which might be represented by this screensnap, making it clear what number N represents and what the total T represents in your story.
2. Draw a picture of your story.
3. Write it as a formula for T .
4. Work out T if $N = 15$.
5. Work out N if $T = 84$.

F

Here is a formula for the total number of something:

$$T = 2N - N$$

1. Draw a screensnap of this.
2. Write a story which it could represent, making it clear what numbers T and N represent in your story.
3. Work out T if $N = 15$.
4. Work out N if $T = 36$.
5. Can you make the formula simpler?
6. Now change your story so that the formula

$$T = 2N + N$$

represents it.

7. Draw the new screensnap to represent this formula.
8. Work out T if $N = 15$.
9. Work out N if $T = 36$.
10. Can you make the formula simpler?

ANNEX II: INTERVIEW SCHEDULES

Pilot study: Year 9G

Initial interviews (held early during the lesson period)

Kerry and Jess, Emma and Felicity, Carly and Holly interviewed.

1. Why did you volunteer to take part in this project [to be one of the students audiotaping their work]?
2. What do you feel about the test last week [the first set of algebra questions]?
(Easy/hard, enough time, format of questions, talking about what things mean).
3. Tell me about the graphic calculators. How are you getting on with them so far?

Do you think they are:

- easy to use
 - fun
 - boring
 - difficult or complicated to use
 - do you understand what the letters mean
 - do you understand the screen commands
 - quicker
4. How do you feel about doing algebra?

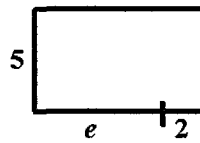
Do you find it hard to remember how to do things?

Is the practice boring?

What do you think the letters mean?

Why do you think we use them sometimes?

5. Review the rectangle question



Which of the following expressions do you think is correct for the area of this rectangle? Tick every one you think is correct.

$5 \times e + 2$, $5 \times (e + 2)$, $10e$, $5 \times e2$, $5(e + 2)$, $e + 2 \times 5$

none correct (give any other answer you have)

6. Discuss methods for solving equations.

What are you actually trying to find out?

Does a different letter mean a different equation?

7. See what they think about

- signs (go with preceding or following term?)
- closure
- negative sign outside a bracket
- $9(3h - 5)$
- $\frac{36ab}{12a}$

8. Do you think equations and expressions are the same thing? Or are they different in some way?

9. Generally, how do you think things are going so far?

Is it too hard, or too boring?

Do you feel you are learning?

10. Is there anything else you want to say or ask?

Follow-up interviews

Gemma and Lauren, and Emma and Felicity were interviewed in pairs.

1. What difference do you think it made, being one of the volunteers?
2. How did using the tape recorder affect your work?
3. Did the awareness you were being taped make any difference, do you think?
4. How did you get on in the tests? Did you find them easy or hard? Was the second easier than the first?
5. How do you feel you got on with the graphic calculators? Do you think they helped or hindered you to learn? Why? Do you think they are fun, boring, difficult, complicated, difficult to understand (abbreviations on the screen), quicker ... Do you think you have changed your mind in any way about using them?
6. How do you feel now about doing algebra? Has your view of doing it changed in any way since before the lessons?
7. What do you think you have learnt?
8. Review simplification of $3x + 2x^2 - 7x - 4x^2$
9. Meaning of equations and expressions. What do you think you are doing when you simplify an expression?
10. What difference to your learning do you think the GC made?

Thanks for volunteering!!

Main case study: Year 7G

Questions based on data provided by each pair, but example given shows the general style of the questions.

Pairs interviewed: Chantelle and Sofia, Nisha and Renelle, Vicki and Georgia, Claire and Briony, Fran and Rebecca, Karen and Nicky, Charlotte and Abigail, Megan and Lucy, Chloe and Sam.

Follow-up interview questions with students

1. *About you*

- Talk about Middle School experiences with algebra: "I have done a bit but I was not very good at it, it confused me."

2. *About the graphic calculator*

- Did the calculator cause any difficulties? Chantelle seemed to be having trouble putting 2.5 into B at the end of the first worksheet.
- There were times when you checked your homework answers by seeing if you both agreed rather than checking with the GC. Were you sure this would be enough? What about $6C - 3B = 6C - B + B + B$ (Chantelle) $= 6 \times C - B + B + B$ (Chantelle) $= 7C - 4B - C - B$ (Sofia).

3. *About Worksheet 1*

- How did you know $AB = BA$?
- You did $2A + 3A$ by working out $2A$ and $3A$ and adding them together. Can you see a different way of doing this?

- Discuss equivalents for $2(A + B)$.
- Did you find this part of the work too easy?

4. *About Expressions Worksheet*

- Is $2A = A \times A$? How do you know?
- In no. 13, Sofia changed $B \times A \times 2 = BAB$ to $2AB$. Why?
- Can you simplify $2A - A$, $3A - A$, $5C - D - C$ now? (They could do $4B - 2B$ at the time).

5. *About the Post-Questionnaire*

- You put $9a + 3b = 12ab$. Explain.
- In answer to question 3, asking if you need to know the value letters represent, one put: "Yes. It can be any number." Does she really think you have to know values?
- Looks like $4D - 3C$ worked out with numbers, $D = 5$, and $C = 4$. Why?
- Similarly with $2W - 3V$, $W = 5$, and $V = 6$.
- Answers to questions 2 and 3, about what letters mean, and whether you need to know their value, one put: "I think a , b and c refer to numbers which have a value of something." And "No". Enquire a little more.
- If $a = 5$, and $b = 9$, what is $2ab$? Didn't know.

Notes for interviews with Year 7 class teachers

In the classroom

- How much time did you spend introducing the students to the GC?
- How quickly do you think they got the hang of it?
- How did you introduce the topic in the first lesson?
- Did you use additional material – Gill certainly did, what was it?
- How would you improve the material for another time?
- What have you done since in algebra with your class?

Types of misconceptions

- Were you aware beforehand of the types of misconceptions that arose – codes, adding symbols next to each other without any operation shown, objects, alphabetic substitution, interpreting letters as arithmetic operations?
- Are there any other types of error you saw which I haven't mentioned?
- Did you realise some students confused \times and $+$, and $/$ and $-$?
- What about $2P = P \times P$ error?
- Did you realise that some were extending the product notation inappropriately into numbers?
- Did you discuss the use of the bracket keys? – errors of the form $2(P + Q)$, do the brackets first, so do $P + Q \times 2$, then GC just confirms their incorrect answer

Theoretical considerations

- Do you feel this approach helped your students to interpret letters as numbers?
- Do you think it helped them to start to learn algebraic syntax?

- What was different from more traditional approaches?
- Do you think that using the GC slowed them down, forcing them to think more about their answers, less likely to be on auto-pilot?
- Do you think that in general if they got something wrong, they tried to work out why, or just marked it wrong, wrote in the right answer, and went on to the next question?
- Do you think this was more or less likely to happen with the GC?
- Do you remember any useful conversations you had in class with individual students or with the class as a whole?
- Has any difference been apparent in lessons done since?
- Do you feel that it leads to an over-numeric approach (always substituting numbers to check something)?
- Using this method, do you think younger children could start to learn algebra?

Future Plans

- Are you teaching a Yr 8 class next year?
- If so, would you be willing to do a similar topic during the autumn term?

Main Case Study: Year 8G

Follow-up interviews with students

Questions for each section put on cards, then placed face down on the table. Students took it in turns to pick up a question, choose an answer, and then continue. Discussion then opened up to the group as a whole.

Introductory questions

- I found the equations [easy/OK/quite hard/very hard] to start with because ...
- I found the equations [easy/OK/quite hard/very hard] by the end of the topic because ...
- I found the graphic calculator [helpful/OK/not very helpful] when I was doing the equations because ...
- When I was doing the equations, I used the graphic calculator to ...

Questions about equations

- An example of an equation is ...
- How would you do these equations? Which is easier and why?
- $\frac{12}{b} = 6$, $\frac{12}{n} = 6$, $\frac{12}{k} = 24$
- How would you do this equation?
 $4(a + 1) = 20$
- How would you do these? Which is easier and why?
 $2.1x - 8.4 = 6.3$, $2x - 8 = 6$
- How would you do these? Which is easier and why?

$$3x - 4 = 2, \quad 3x - 4 = 2x + 6$$

- How would you do these? Which is easier and why?

$$6x + 3 = 3x + 6, \quad 6 + 3x = 3 + 6x$$

- How would you do this? Do you get a sensible answer?

$$9x + 8 = 3x + 8$$

Teacher interview questions

General questions

- Did they find it easy to solve the equations to start with?
- How about later on – did they find it got difficult?
- What do they think an equation is? Can they distinguish an equation from an expression?
- Do you think the GC was helpful or a nuisance for them?
- How did you tell them to use it?
- How do they check their answers? Do they see why they might want to check their answers?

Easy equation questions

- How do you think they would handle:

$$\frac{12}{n} = 6$$

Is $\frac{12}{p} = 6$ the same as $\frac{12}{m} = 6$

$$\frac{12}{n} = 24$$

$$4(a + 1) = 20$$

Harder equation questions

- How do you think they would do these - would the first help with the second?

$$2x - 8 = 6 \text{ and } 2.1x - 8.4 = 6.3$$

$$3x - 4 = 2 \text{ and } 3x - 4 = 2x + 6$$

$$6x + 3 = 3x + 6 \text{ and } 6 + 3x = 3 + 6x$$

$$9x + 8 = 3x + 8 \text{ - is } x = 0 \text{ a sensible answer to them?}$$

Methods

- What kinds of methods were you aware of them using to solve the equations?
- How much do you think was guess-and-check, how much appropriate use of procedural or holistic methods?
- What worksheets did you use?
- How did you teach them to solve equations?

Survey: Years 6-8

Teacher Questionnaire

Name: School:.....

Type of school: Years taught (6, 7 or 8):

Sets taught: No. of sets in a year:

1. How would you describe this/these class(es)' general mathematical ability?
2. What previous algebra experience have they had to your knowledge?
3. How would you rate their understanding of the way letters are used in algebra before starting this topic?
4. How would you rate their understanding now?
5. How did they get on with the graphic calculators?
6. How much do you think the model of a variable given by the graphic calculator helped them?
7. Would you be prepared to use this model again?
8. Would you be prepared to use these worksheets again? What changes would you like to see?
9. What additional materials did you use? (Please describe or include photocopies)
10. Did you do anything significantly different from the procedure in the general instructions and the lesson plans (I just need to know!)
11. Anything else you would like to add

ANNEX III: QUESTIONNAIRES

Pilot study: Year 9G

Initial questionnaire

Name

1. How well do you think you did on the test on Monday? Very well

OK

Not very well

Which part do you think you did best?

About letters Qu.1)

Equations (Qu.2)

Expressions (Qu.3)

Which do you think you did least well?

About letters

Equations

Expressions

Did you have enough time to complete it?

Yes

No

2. Comparing algebra with other Mathematics topics

More enjoyable?

you have done, do you think it is

About the same

Less enjoyable?

Other (explain below)

3. Why? (Explain your response to question 5).

4. Do you like using the graphic calculators?
- They're good
- OK
- Boring
- Hard to use
- Other (explain below)
5. Why? (Explain your response to question 7).
6. How do you think you get on at Mathematics generally?
- Good
- OK
- Not very well
7. How much do you like Mathematics compared with other subjects?
- Good
- OK
- Not very much
8. What makes it good or not?

Initial algebra test**Question 1**

For this question, just tick every answer you think is correct.

(i) What does $5y$ mean?

$5 + y$

5 and y

$5 \times y$

$5 + 5 + 5 + 5 + 5$

$y + y + y + y + y$

other answer (please write)

(ii) What does a^2 mean?

$a \times 2$

$a + 2$

$2 \times a$

$2 + 2$

$a + a$

a and 2

other answer (please write)

(iii) What does bc mean?

b and c

$b \times c$

$b + c$

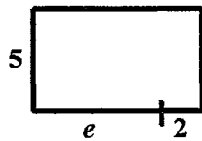
$2 + 3$

2×3

other answer (please write)

Answer these questions in the space given:

- (iv) If $a = 5$, what does $6a$ mean?
- (v) If $b = 9$, what does $b4$ mean?
- (vi) Which of the following expressions do you think is correct for the area of this rectangle? Tick every one you think is correct.



$$5 \times e + 2$$

$$5 \times (e + 2)$$

$$10e$$

$$5 \times e2$$

$$5(e + 2)$$

$$e + 2 \times 5$$

none correct (give any other answer you have)

Question 2

Please answer each question in the space given in the left hand column. In the corresponding space in the right hand column, explain how you went about answering the question. Give as much detail here as you can, and don't worry about whether you are using the "right" method or not – I want to see what you *actually* do, not what you think I want you to do!

Solve the following equations:

- (i) $3n - 98 = 115$

(ii) $11(n + 63) = 264$

(iii) $11n + 14n = 375$

(iv) $5n + 17 = 2n + 61$

(v) $29 - 3n = -49$

(vi) $\frac{7n}{15} - 71 = 55$

(vii) $\frac{7a}{15} - 71 = 55$

(viii) $\frac{36}{n} + 137 = 149$

Question 3

For these questions, ring the letter corresponding to any expressions you think are equivalent to (mean the same as) the one given. If you don't think any of them give an equivalent expression, write that in your explanation. Use the left hand column for any working out you may wish to do, and the right hand column to explain how you came to your answer.

(i) Add $2a$ onto $5a$

<i>A</i>	$7aa$	<i>B</i>	$7a$	<i>C</i>	7	<i>D</i>	$10a$
----------	-------	----------	------	----------	-----	----------	-------

(ii) Add 4 onto $8n$

<i>A</i>	$12n$	<i>B</i>	12	<i>C</i>	$4 + 8n$	<i>D</i>	$8n + 4$
----------	-------	----------	------	----------	----------	----------	----------

(iii) Multiply $3 + b$ by 9

<i>A</i>	$27 + 9b$	<i>B</i>	$9b + 27$	<i>C</i>	$27 + b$	<i>D</i>	$b + 27$
----------	-----------	----------	-----------	----------	----------	----------	----------

(iv) $2a + 5b$

A	$10ab$	B	$7ab$	C	7	D	25
-----	--------	-----	-------	-----	-----	-----	------

(v) $8n - 9m - 6n + 4m$

A	$14n - 5m$	B	$2n - 5m$	C	$2n + 13m$	D	$-5m - 2n$
-----	------------	-----	-----------	-----	------------	-----	------------

(vi) $9(3h - 5)$

A	$27h - 5$	B	$-45 + 27h$	C	$27h - 14$	D	$5 - 27h$
-----	-----------	-----	-------------	-----	------------	-----	-----------

(vii) $-(9r - 3s)$

A	$9r - 3s$	B	$-9r - 3s$	C	$9r + 3s$	D	$-9r + 3s$
-----	-----------	-----	------------	-----	-----------	-----	------------

(viii) $\frac{36ab}{12a}$

A	$3ab$	B	$3a$	C	$3b$	D	3
-----	-------	-----	------	-----	------	-----	-----

Follow-up questionnaire

Name

- | | |
|--|--|
| 1. How well do you think you did
on the test today? | Very well
OK
Not very well |
| 2. Do you think you did better
than on the test two weeks ago? | Yes
No
About the same |
| 3. Which part do you think you did best? | About letters (Qu.1)
Equations (Qu.2)
Expressions (Qu.3)
Simultaneous Equations (Qu. 4) |
| 4. Which do you think you did least well? | About letters
Equations
Expressions
Simultaneous equations |
| 5. Did you have enough time to complete it? | Yes
No |
| 5. Comparing algebra in this topic
with what you expected it to be like,
have you found it to be ... | More enjoyable?
About what you expected
Less enjoyable?
Other (explain below) |
| 6. Why? (Explain your response to question 6). | |

7. How has this topic compared with others
you have done recently, e.g. trigonometry,
as far as difficulty is concerned?
- Less difficult
About the same
Harder
8. Why? (Explain your response to question 8).
9. Have you enjoyed using the graphic calculators?
- Yes
It was OK
No – boring
No – hard to use
Other (explain below)
10. Why? (Explain your response to question 10).
11. Do you think you have improved at algebra?
- Yes
Stayed the same
No
12. In what ways have you improved, or not?
13. Which aspect of the work have you enjoyed most and least?
- (Consider work on equations and expressions, and also whether you prefer doing
screensnaps, working on exercises written algebraically as in a text book, or problems
from the board).
14. If you used a tape recorder,
how much difference do you think this made?
- A lot
Not very much
None at all
15. If you think it made a difference, what difference do you think it made and why?

Post-topic Algebra Test***Question 1***

For this question, just tick every answer you think is correct.

(i) What does $3k$ mean?

$$3 \times k$$

$$k + k + k$$

$$3 + k$$

$$3 + 3 + 3$$

$$3 \text{ and } k$$

other answer (please write)

(ii) What does $m4$ mean?

$$m + 4$$

$$m \times 4$$

$$4 \times m$$

$$4 + 4 + 4 + 4$$

$$m + m + m + m$$

$$m + m$$

$$m \text{ and } 4$$

other answer (please write)

(iii) What does ec mean?

$$e \text{ and } c$$

$$e \times c$$

$$e + c$$

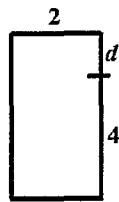
$$5 + 3$$

$$5 \times 3$$

$$3 \times 5$$

other answer (please write)

- (iv) If $p = 8$, what does $9p$ mean?
- (v) If $f = 5$, what does $f7$ mean?
- (vi) Which of the following expressions do you think is correct for the area of this rectangle? Tick every one you think is correct.



$$2 \times d + 4$$

$$2 \times (d + 4)$$

$$8d$$

$$2 \times d4$$

$$2 (d + 4)$$

$$d + 4 \times 2$$

give any other answer you have

Question 2

Please answer each question in the space given in the left hand column. In the corresponding space in the right hand column, either show your working, or explain how you went about answering the question.

Solve the following equations:

(i) $9g + 187 = 133$

(ii) $207 (k - 92) = 3933$

(iii) $54r - 39r = 840$

(iv) $17d - 89 = 11d + 67$

(v) $83 - 3w = 503$

(vi) $\frac{5p}{19} + 78 = 56$

(vii) $\frac{18}{2q} - 45 = -63$

(viii) $\frac{5p}{19} + 78 = 56$

Question 3

For these questions, ring the letter corresponding to any expressions you think are equivalent to (mean the same as) the one given. If you don't think any of them give an equivalent expression, say so in the space given.

(i) Add $7s$ onto $6s$

A $13ss$ B $13s$ C 13 D $42s$

(ii) Add 3 onto $9v$

A $12v$ B 12 C $3 + 9v$ D $9v + 3$

(iii) Multiply $9 + c$ by 5

A $45 + 5c$ B $5c + 45$ C $45 + c$ D $c + 45$

(iv) $7m + 6y$

A $13my$ B $42my$ C 13 D 42

(v) $3(8h - 6)$

$A \quad 24h - 6 \quad B \quad -18 + 24h \quad C \quad 38h - 6 \quad D \quad -18 + 24h$

(vi) $2c - 5j - c + 9j$

$A \quad c - 14j \quad B \quad 3c + 4j \quad C \quad c + 4j \quad D \quad -14j - 3c$

(vii) $-(2z - 5q)$

$A \quad 2z - 5q \quad B \quad -2z - 5q \quad C \quad -2z + 5q \quad D \quad -5q + 2z$

(viii) $\frac{33ba}{3b}$

$A \quad 11ba \quad B \quad 11b \quad C \quad 11a \quad D \quad 11$

Main case study: Year 7G

Pre-Questionnaire

Name: Form:

1. What middle school did you go to?
2. If you were put in sets or special groups for Mathematics, which were you in?
3. Have you ever done a topic called algebra, or had anything about algebra explained to you before? If your answer is yes, please tell me as much as possible about it.
4. Have you ever used a graphic calculator before (one with a big screen that you can draw graphs on, not an ordinary calculator)? If your answer is yes, tell me as much as possible about it.
5. What do you think the answers to these questions are?

(If you have no idea just say so, but have a guess if you can, and say it's a guess.)

(a) $6a + 2a = \dots\dots\dots$

(b) $12b - 2b = \dots\dots\dots$

(c) $6a + 2b = \dots\dots\dots$

6. What do you think the a and b in question 5 refer to?

7. Can you solve this equation? $3x - 12 = 24$

(If you have no idea what to do, just say so, but have a guess if you can, and say it's a guess.)

8. How did you work out your answer to question 7?

Post-questionnaire

Name: Form:.....

1. What do you think the answers to these questions are?

(If you have no idea just say so, but have a guess if you can, and say it's a guess.)

(a) $8a + 6a =$

(b) $15c - 7c =$

(c) $9a + 3b =$

2. What do you think the a , b and c in question 1 refer to?

3. Do you have to know values for a , b and c to answer question 1? If your answer is yes, say what you think they might be.

4. Can you solve this equation? $5x + 20 = 45$

(If you have no idea what to do, just say so, but have a guess if you can, and say it's a guess.)

5. How did you work out your answer to question 3?

6. Which of the following expressions do you think are the same as $35ab$? (There may be several – ring each one you think may be the same).

A $5 \times 3 \times b \times a$

B $a \times b \times 35$

C $7 \times 5 \times a \times b$

D $b \times 35 \times a$

E $b \times 7 \times 5 \times a$

F $b \times 5 \times 3 \times a$

7. If $a = 5$ and $b = 9$, which of the following are correct values of $2ab$?

A 259

B 90

C 180

8. Here is a student's work (none of yours!). I want you to mark it. If you think she is right, just tick the question; if you think she is wrong, mark it with a cross, and put what you think the right answer should be.

(a) $C + D = D + C$

(b) $S \times T = TS$

(c) $2Y = Y \times Y$

(d) $3R - S = R + R + R - S$

(e) $2(A + D) = 2A + 2D$

(f) $4D - 3C = 2D \times 2D - 3C$

(g) $2W - 3V = W + W - V + V + V$

(h) $2B - B = B$

Main case study: Year 8G

Pre-Questionnaire

Name:

1. Write these expressions as simply as possible:

(a) $3a + 5a =$

(b) $\frac{2a}{2a} =$

(c) $2b - b =$

(d) $5b - 2a + 7b =$

What do you think a and b stand for?

2. Circle each of the following expressions which mean the same as $24pq$:

$2 \times 4 \times p \times q$

$p \times 24 \times q$

$q \times 42p$

$pq24$

$p \times q \times 3 \times 8$

$pq64$

$2p4q$

$q \times p \times 6 \times 4$

3. What is x in each case?

(a) $4x - 36 = 12$

(b) $2x + 4x = 66$

How did you work out your answers?

4. Look at the table. What can you say about the relationship between x and y ?

x	1	2	3	4	5
y	3	5	7	9	11

5. (a) What can you say about m , if $m = 5n + 2$ and $n = 4$?

(b) What can you say about b , if $b = c + f$ and $b + c + f = 30$?

6. (a) If $a + b = 57$, then $a + b + 2 =$

(b) Add 4 onto $n + 43$

(c) If $n - 246 = 762$, then $n - 247 =$

(d) If $e + f = 8$, then $e + f + g =$

(e) Add 4 onto $7n$

7. (a) What can you say about c , if $c + d = 10$ and c is less than d ?

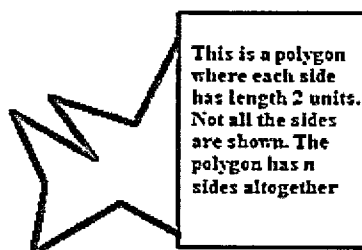
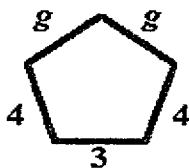
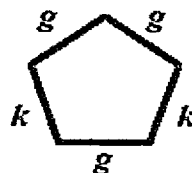
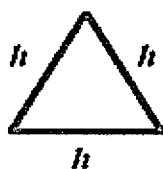
(b) $a + b + c = a + p + c$

Is this true: always / sometimes / never?

Ring the answer you think is correct, and explain your answer.

8. Which is larger, $2n$ or $n + 2$? Explain your answer.

9. Write down an expression for the perimeter (distance around the outside) of each of the following shapes:



Post-Questionnaire

Name:

1. Tick the box which best describes how you felt when you did the first questionnaire.

The first questionnaire we did was:

very easy ☐ quite easy ☐ OK ☐ quite hard ☐ very hard ☐

2. Look at the table. What can you say about the relationship between x and y ?

x	1	2	3	4	5
y	4	7	10	13	16

3. Write these expressions as simply as possible:

(a) $6b + 4b = \dots\dots\dots$

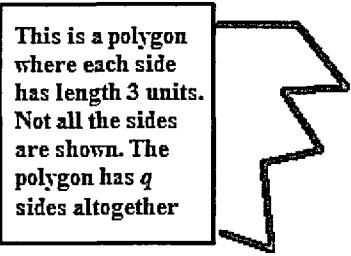
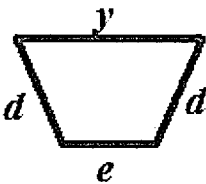
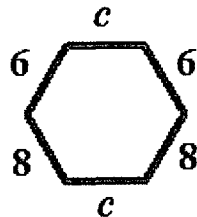
(b) $2a - a = \dots\dots\dots$

(c) $\frac{4a}{4a} = \dots\dots\dots$

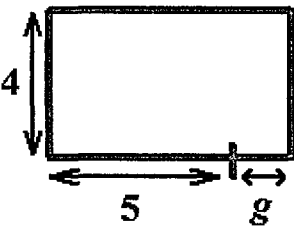
(d) $3a - b + 6a = \dots\dots\dots$

What do you think a and b stand for?

4. Write down an expression for the **perimeter** (distance around the outside) of each of the first three shapes. For the fourth shape, write down an expression for its **area**.



Write down the **area** of this shape:



5. Which is larger, $p + 3$ or $3p$? Explain your answer.
6. (a) What can you say about g , if $g = 6d + 1$ and $d = 5$?
- (b) What can you say about r , if $r = p + w$ and $r + p + w = 18$?
7. (a) If $a + b = 79$, then $a + b + 3 =$
- (b) Add 7 onto $52 + p$
- (c) If $v - 583 = 895$, then $v - 584 =$
- (d) Add 7 onto $5b$
8. If $y + w = 6$, then $y + w + x =$
9. Circle each of the following expressions which mean the same as $36bc$:
- | | | | |
|--------------------------------|--------------------------------|--------------------------------|----------------|
| $cb49$ | $c \times 36 \times b$ | $cb36$ | $3c6b$ |
| $c \times 4 \times b \times 9$ | $b \times c \times 6 \times 6$ | $3 \times 6 \times c \times b$ | $b \times 63c$ |

10. What is x in each case?

(d) $3x - 11 = 25$

(e) $3x + 2x = 60$

(f) $7x - 12 = 3x + 4$

How did you work out your answers?

11. (a) What can you say about q , if $p + q = 12$ and q is less than p ?

(b) $a + b = b$

Is this true: always / sometimes / never?

Ring the answer you think is correct, and explain your answer.

12. Tick the box which best describes how you feel about this questionnaire. This

questionnaire was:

very easy ☐ quite easy ☐ OK ☐ quite hard ☐ very hard ☐

13. Compared to the first questionnaire, this one was:

easier ☐ about the same ☐ harder ☐

Survey: Years 6-8

Pre-Questionnaire

A You and your school

Name: Date:.....

School:

School year (6, 7 or 8): Your teacher:.....

Age: yrs mths

Sex: M ☐ F ☐

B Previous experience of algebra and graphic calculators

1. Have you ever done a topic called algebra, or had anything about algebra explained to you before?

No ☐ A little (not more than a couple of lessons) ☐ Yes (more than a couple of lessons) ☐

2. Have you ever used a graphic calculator before (one with a big screen that you can draw graphs on, not an ordinary calculator)?

No ☐ A little (not more than a couple of times) ☐ Yes (more than a couple of times) ☐

C *Now try some algebra questions – don't worry if you haven't done questions like these before, this is not a test! Just write down what you think the answers might be.*

3. $6a + 2a =$

4. $12b - 2b =$

5. $4a + 3b + 2a =$

6. $5a - 2b + 7a = \dots\dots\dots$
7. What do you think the a and b mean?
8. What do you think j stands for here? $2j - 6 = 18$
9. How did you work out your answer to question 5?

D *In this section ring all the answers which you think are right (there could be more than one).*

10. Ring anything you think is the same as $2a$:

$2 + a$ $2 \times a$ 2 $a + a$ 21 a^2

11. If $b = 3$ and $c = 5$, what do you think bc is? Ring any answer you agree with:

23 35 15 2 8

If you think it is something else, please write it down

12. Which of the following expressions do you think are the same as $12pq$? Ring any answer you agree with.

$6 \times 2 \times p \times q$ $p \times q \times 21$ $1 \times 2 \times p \times q$ $q \times 12 \times p$ $12 \times p \times q$

Immediate Post-Questionnaire**A You and your school**

Name: Date:

School:

B How did you get on with the graphic calculator, and the algebra lessons?

1. How helpful did you find the graphic calculator?

Very helpful ☐ Quite helpful ☐ OK ☐ Not very helpful ☐ Not at all helpful ☐

2. How did you get on with the algebra?

Very easy ☐ Quite easy ☐ OK ☐ Quite hard ☐ Very hard ☐**C Try these algebra questions – remember this is NOT a test! Just put down what you think the answers are.**

3. $8b + 4b = \dots\dots\dots$

4. $13a - 9a = \dots\dots\dots$

5. $6c + 2a + 3c = \dots\dots\dots$

6. $8a - 5c + 2a = \dots\dots\dots$

7. What do you think the a and c mean?8. What do you think f stands for here? $4f - 30 = 14$

9. How did you work out your answer to question 5?

D In this section ring all the answers which you think are right (there could be more than one).

10. Ring anything you think is the same as $2b$:

8 $2 \times b$ $2 + b$ 2 22 $b + b$ $b \times b$

11. If $a = 2$ and $c = 6$, what do you think ac is? Ring any answer you agree with:

26 3 12 4 8

If you think it is something else, please write it down

12. Which of the following expressions do you think are the same as $36ds$? Ring any answer you agree with.

$d \times s \times 63$ $3 \times 6 \times d \times s$ $d \times 36 \times s$ $36 \times d \times s$ $6 \times 6 \times d \times s$

Delayed Post-Questionnaire

A You and your school

Name: Date:

School:

B Try these algebra questions – remember this is NOT a test! Just put down what you think the answers are.

1. $8b + 4b = \dots\dots\dots$

2. $13a - 9a = \dots\dots\dots$

3. $4b + 7a + 3b = \dots\dots\dots$

4. $7a - 3b + 4a = \dots\dots\dots$

5. What do you think the a and b mean?

6. What do you think f stands for here? $4f - 30 = 14$

7. How did you work out your answer to question 4?

C In this section ring all the answers which you think are right (there could be more than one).

8. Ring anything you think is the same as $2b$:

8 $2 \times b$ $2 + b$ 22 $b + b$ $b \times b$ 2

9. If $a = 2$ and $c = 6$, what do you think ac is? Ring any answer you agree with:

26 3 12 4 8

If you think it is something else, please write it down

10. Which of the following expressions do you think are the same as $36ds$? Ring any answer you agree with.

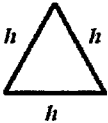
$$d \times s \times 63 \quad 36 \times d \times s \quad 3 \times 6 \times d \times s \quad s \times 36 \times d \quad 6 \times 6 \times d \times s$$

ANNEX IV: QUESTIONS USED FOR COMPARISON WITH KÜCHEMANN’S ANALYSIS

Level 1

In Table 17, questions are shown which were considered to be at Küchemann’s level 1, that is, those which can be answered without interpreting the letters as numbers, either by ignoring the letters, treating them as objects, or substituting specific numbers.

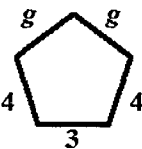
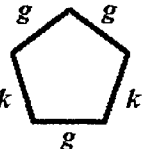
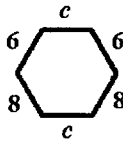
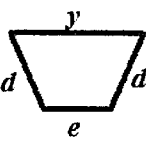
Table 17: Questions from the various classroom studies considered to be at Küchemann's level 1

Classroom Study	No. of questions	Pre-Questionnaire	Immediate Post-Questionnaire	Delayed Post-Qu'naire
Year 9 Pilot Study	3	If $a = 5$, what does $6a$ mean? If $b = 9$, what does $b4$ mean? Add $2a$ onto $5a$ (options: $7aa$, $7a$, 7 , $10a$)	If $g = 6$, what does $6g$ mean? If $u = 8$, what does $u5$ mean? Add $4p$ onto $5p$ (options: $9pp$, $9p$, 9 , $20p$)	
Year 7 Case Study	1	$6a + 2a$	$8a + 6a$	
Year 8 Case Study	3/2	$3a + 5a$ If $a + b = 57$, then $a + b + 2 =$ Write down an expression for the perimeter (distance around the outside) of the following shape: 	$6b + 4b$ If $a + b = 79$, then $a + b + 3 =$	
Year 6-8 Survey	1	$6a + 2a =$	$8b + 4b =$	$8b + 4b =$

Level 2

In Table 18, questions are shown which were considered to be at Küchemann's level 2, that is, those which can be answered without interpreting the letters as numbers, either by ignoring the letters, treating them as objects, or substituting specific numbers, but are structurally more complex than level 1 questions.

Table 18: Questions from the various classroom studies considered to be at Küchemann's level 2


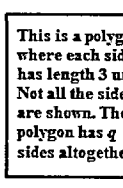
Classroom Study	No. of questions	Pre-Questionnaire	Immediate Post-Questionnaire	Delayed Post-Questionnaire
Year 9 Pilot Study	5/3 (Any correct option accepted) (Any correct option accepted)	What does $5y$ mean? (Options: $5 + y$, 5 and y , $5 \times y$, $5 + 5 + 5 + 5 + 5$, $y + y + y + y + y$, other answer) What does $a2$ mean? (Options: $a \times 2$, $a + 2$, $2 \times a$, $2 + 2$, $a + a$, a and 2 , other answer)	What does $7c$ mean? (Options: $7 + c$, 7 and c , $7 \times c$, $7 + 7 + 7 + 7 + 7$, $c + c + c + c + c$, other answer) What does $w4$ mean? (Options: $w \times 4$, $w + 4$, $4 \times w$, $4 + 4$, $w + w$, w and 4 , other answer)	
Year 7 Case Study	1	$12b - 2b$	$15c - 7c$	
Year 8 Case Study	5	<p>$2b - b$</p> <p>What can you say about m, if $m = 5n + 2$ and $n = 4$?</p> <p>Write down an expression for the perimeter (distance around the outside) of the following shape:</p>  <p>Write down an expression for the perimeter (distance around the outside) of the following shape:</p>  <p>Add 4 onto $n + 43$</p>	<p>$2a - a$</p> <p>What can you say about g, if $g = 6d + 1$ and $d = 5$?</p> <p>Write down an expression for the perimeter (distance around the outside) of the following shape:</p>  <p>Write down an expression for the perimeter (distance around the outside) of the following shape:</p>  <p>Add 7 onto $52 + p$</p>	

<i>Classroom Study</i>	<i>No. of questions</i>	<i>Pre-Questionnaire</i>	<i>Immediate Post-Questionnaire</i>	<i>Delayed Post-Questionnaire</i>
<i>Year 6-8 Survey</i>	3 (Any correct options on 3 rd question accepted)	$12b - 2b$ $4a + 3b + 2a$ Ring anything you think is the same as $2a$ (Options: $2 + a$, $2 \times a$, 2 , $a + a$, 21 , $a2$)	$13a - 9a$ $6c + 2a + 3c$ Ring anything you think is the same as $2b$ (Options: 8 , $2 \times b$, $2 + b$, 2 , 22 , $b + b$, $b \times b$)	$13a - 9a$ $4b + 7a + 3b$ Ring anything you think is the same as $2b$ (Options: 8 , $2 \times b$, $2 + b$, 22 , $b + b$, $b \times b$, 2)

Level 3

In Table 19, questions are shown which were considered to be at Küchemann's level 3, that is, questions which require letters to be interpreted as a specific unknown.

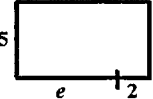
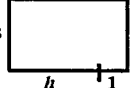
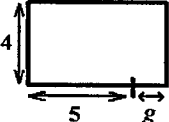
Table 19: Questions from the various classroom studies considered to be at Küchemann's level 3

Class-room Study	No. of questions	Pre-Questionnaire	Immediate Post-Questionnaire	Delayed Post-Qu'naire
Year 9 Pilot Study	3 (1 st qu. has 2 correct options) (Correct response to 3 rd qu. is to choose none.)	Add 4 onto $8n$. (Options: $12n$, 12 , $4 + 8n$, $8n + 4$) $2a + 5b$ (Options: $10ab$, $7ab$, 7 , 25)	Add 7 onto $4s$. (Options: $28s$, 28 , $7 + 4s$, $4s + 7$) $3d + 6c$ (Options: $18dc$, $9dc$, 9 , 36)	
Year 7 Case Study	1	$6a + 2b$	$9a + 3b$	
Year 8 Case Study	6	$5b - 2a + 7b$ What can you say about b , if $b = c + f$ and $b + c + f = 30$? If $e + f = g$, then $e + f + g =$ Add 4 onto $7n$ What can you say about c , if $c + d = 10$ and c is less than d ? (Answer: 1, 2, 3, 4) Write down an expression for the perimeter (distance around the outside) of the following shape: <div data-bbox="521 1382 786 1570">  <p>This is a polygon where each side has length 2 units. Not all the sides are shown. The polygon has n sides altogether</p> </div>	$3a - b + 6a$ What can you say about r , if $r = p + w$ and $r + p + w = 18$? If $y + w = 6$, then $y + w + x =$ Add 7 onto $5b$ What can you say about q , if $p + q = 12$ and q is less than p ? (Answer 1, 2, 3, 4, 5) Write down an expression for the perimeter (distance around the outside) of the following shape: <div data-bbox="899 1382 1156 1570">  <p>This is a polygon where each side has length 3 units. Not all the sides are shown. The polygon has q sides altogether</p> </div>	
Year 6-8 Survey	1	$5a - 2b + 7a$	$8a - 5c + 2a$	$7a - 3b + 4a$

Level 4

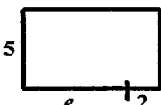
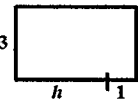
In Table 20, questions are shown which were considered to be at Küchemann's level 4, that is, questions which require an understanding of letters as at least that of a specific unknown and possibly as a variable; also structurally more complex than level 3.

Table 20: Questions from the various classroom studies considered to be at Küchemann's level 4 (none on Y7 case study or Y6-8 survey)

Class-room Study	No. of questions)	Pre-Questionnaire	Immediate Post-Questionnaire
Year 9 Pilot Study	7 (Two options correct on 1 st question) (Two options correct on 2 nd question)	<p>Which of the following expressions do you think is correct for the area of this rectangle? Tick every one you think is correct.</p>  <p>(Options: $5 \times e + 2$, $5 \times (e + 2)$, $10e$, $5 \times e2$, $5(e + 2)$, $e + 2 \times 5$, none correct/other answer)</p> <p>Multiply $3 + b$ by 9 (Options: $27 + 9b$, $9b + 27$, $27 + b$, $b + 27$)</p> <p>$8n - 9m - 6n + 4m$ (Options: $14n - 5m$, $2n - 5m$, $2n + 13m$, $-5m - 2n$)</p> <p>$9(3h - 5)$ (Options: $27h - 5$, $-45 + 27h$, $27h - 14$, $5 - 27h$)</p> <p>$-(9r - 3s)$ (Options: $9r - 3s$, $-9r - 3s$, $9r + 3s$, $-9r + 3s$)</p>	<p>Which of the following expressions do you think is correct for the area of this rectangle? Tick every one you think is correct.</p>  <p>(Options: $3 \times h + 1$, $3 \times (h + 1)$, $4h$, $3 \times h1$, $3(h + 1)$, $h + 1 \times 3$, none correct/other answer)</p> <p>Multiply $4 + r$ by 8 (Options: $32 + 8r$, $8r + 32$, $32 + r$, $r + 32$)</p> <p>$5c - 8a$ $3c + 9a$ (Options: $8c - 17a$, $2c + a$, $2c + 17a$, $-17a - 2c$)</p> <p>$7(5y - 2)$ (Options: $35y - 7$, $-14 + 35y$, $35y - 14$, $7 - 35y$)</p> <p>$-(7p - 4q)$ (Options: $7p - 4q$, $-7p - 4q$, $7p + 4q$, $-7p + 4q$)</p>
Year 8 Case Study	¾	<p>What can you say about c, if $c + d = 10$ and c is less than d? (Answer: an interval bounded above by 4, 4.9 or 5)</p> <p>$a + b + c = a + p + c$ Is this true: always/sometimes/never Ring the answer you think is correct, and explain your answer.</p> <p>Which is larger, $2n$ or $n + 2$? Explain your answer.</p>	<p>What can you say about q, if $p + q = 12$ and q is less than p? (Answer: an interval bounded above by 5, 5.9 or 6)</p> <p>$a + b = b$ Is this true: always/sometimes/never Ring the answer you think is correct, and explain your answer.</p> <p>Which is larger, $3n$ or $n + 3$? Explain your answer. Write down an expression for this area:</p> 

ANNEX V: QUESTIONS USED FOR ANALYSIS OF PROCEPTUAL ANALYSIS

Table 21: Questions from the various classroom studies considered to require proceptual thinking to be answered successfully

Classroom Study	No. of questions	Pre-Questionnaire	Immediate Post-Questionnaire	Delayed Post-Questionnaire																								
Year 9 Pilot Study	5 (Two correct options in 1 st question) (Two correct options in 2 nd question)	Which of the following expressions do you think is correct for the area of this rectangle? Tick every one you think is correct.  (Options: $5 \times e + 2$, $5 \times (e + 2)$, $10e$, $5 \times e2$, $5(e + 2)$, $e + 2 \times 5$, none correct/other answer) Add 4 onto $8n$. (Options: $12n$, 12 , $4 + 8n$, $8n + 4$) $36ab/12a$ (Options: $3ab$, $3a$, $3b$, 3)	Which of the following expressions do you think is correct for the area of this rectangle? Tick every one you think is correct.  (Options: $3 \times h + 1$, $3 \times (h + 1)$, $4h$, $3 \times h1$, $3(h + 1)$, $h + 1 \times 3$, none correct/other answer) Add 7 onto $4s$. (Options: $28s$, 28 , $7 + 4s$, $4s + 7$) $48fe/6f$ (Options: $8fe$, $8f$, $8e$, 8)																									
Year 8 Case Study	6	$2a/2a$ Look at the table. What can you say about the relationship between x and y . <table border="1" data-bbox="550 1254 817 1313"><tr><td>x</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td></tr><tr><td>y</td><td>3</td><td>5</td><td>7</td><td>9</td><td>11</td></tr></table> (Answer relating x and y correctly required) What can you say about b , if $b = c + f$ and $b + c + f = 30$? If $n - 246 = 762$, then $n - 247 =$ (Answer given correctly without evaluating n) If $e + f = g$, then $e + f + g =$ Add 4 onto $7n$	x	1	2	3	4	5	y	3	5	7	9	11	$4a/4a$ Look at the table. What can you say about the relationship between x and y . <table border="1" data-bbox="878 1254 1155 1313"><tr><td>x</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td></tr><tr><td>y</td><td>4</td><td>7</td><td>10</td><td>13</td><td>16</td></tr></table> (Answer relating x and y correctly required) What can you say about r , if $r = p + w$ and $r + p + w = 18$? If $v - 538 = 895$, then $v - 584 =$ (Answer given correctly without evaluating v) If $y + w = 6$, then $y + w + x =$ Add 7 onto $5b$	x	1	2	3	4	5	y	4	7	10	13	16	
x	1	2	3	4	5																							
y	3	5	7	9	11																							
x	1	2	3	4	5																							
y	4	7	10	13	16																							
Year 6-8 Survey	2	$4a + 3b + 2a$ $5a - 2b + 7a$	$6c + 2a + 3c$ $8a - 5c + 2a$	$4b + 7a + 3b$ $7a - 3b + 4a$																								

ANNEX VI: FULL RESULTS OF ALL ANALYSES

Full results by year group

Case Studies (Phase I), school G

Statistically significant improvements are shaded in pink.

Table 22: Results of all analyses for case studies

Analysis	Year group, and questionnaire	n	Mean	SD	p, if less than 0.05
Mean correct score	7G(1)	78	31.41	29.176	6.17×10^{-6}
	7G(2)	79	51.266	27.706	
	8G(1)	27	56.397	16.973	
	8G(2)	28	62.013	14.438	
	9G(1)	30	67.857	17.471	0.00558
	9G(2)	30	78.151	13.729	
Mean Level on Küchemann's Framework	7G(1)	78	0.5897	0.7105	2.24×10^{-8}
	7G(2)	79	1.3797	1.0658	
	8G(1)	27	2(2)222	1.086	
	8G(2)	28	2.6429	0.9894	
	9G(1)	30	2.667	1.0743	0.0363
	9G(2)	30	3.3333	0.8023	
% students showing proceptual understanding	8G(1)	27	25.926	44.658	
	8G(2)	28	53.571	50.787	
	9G(1)	30	90	30.513	
	9G(2)	30	100	0	

Survey (Phase II), schools A, B, C and D

Statistically significant improvements are shaded in pink where showing an improvement, and in blue where showing a deterioration.

Table 23: Full results for Year 6-8 survey

Analysis	Year group, and questionnaire	n	Mean	SD	p, if less than 0.05
Mean correct score	6(1)	36	34.491	21.561	
	6(2)	35	42.208	19.646	
	6(3)	32	42.33	18.56	
	7(1)	124	49.093	21.441	
	7(2)	125	52.509	23.502	
	7(3)	125	53.127	21.1	
	8(1)	119	52.976	21.295	
	8(2)	112	54.058	21.678	
	8(3)	126	50.902	21.57	
Mean level on Küchemann's framework	6(1)	36	1.0556	1.0405	0.0133 for pre to post
	6(2)	35	1.5714	0.9167	
	6(3)	32	1.3125	0.8958	
	7(1)	124	1.6613	1.0272	
	7(2)	125	1.688	1.0505	
	7(3)	125	1.776	0.8786	
	8(1)	119	2.672	0.9273	
	8(2)	112	1.9107	1.0183	
	8(3)	126	1.7619	0.9669	0.00582 for pre to delayed post
% students showing proceptual understanding	6(1)	36	13.889	35.074	
	6(2)	35	22.857	42.604	
	6(3)	32	12.5	33.601	
	7(1)	124	44.355	49.882	
	7(2)	125	43(2)	49.735	
	7(3)	125	46.4	50.071	
	8(1)	119	65.546	47.723	
	8(2)	112	65.179	47.855	
	8(3)	126	60.317	49.119	

Full results by initial level on Küchemann’s framework

Level 1 or below

Level 1 corresponds to an interpretation of a letter as: letter ignored, letter interpreted as object, or letter evaluated.

Table 24: Full results for students initially at Küchemann’s level 1 or below

Analysis	Questionnaire	n	Mean	SD	p, if less than 0.05
Mean correct score	1	183	29.52	21.346	4.58×10^{-9}
	2	171	43.122	23.075	
Mean level on Küchemann’s framework	1	183	0.52459	0.50077	0 (z = -7.8734)
	2	171	1.18713	0.9882	
% students showing proceptual understanding	1	183	3.825	19.233	0.00104
	2	171	12.865	33.58	

Level 2

Understanding of a letter as in level 1, but able to cope with structurally more difficult questions, and some lack of closure.

Table 25: Full results for students initially at Küchemann’s level 2

Analysis	Questionnaire	n	Mean	SD	p, if less than 0.05
Mean correct score	1	122	53.101	13.676	
	2	113	55.108	13.754	
Mean level on Küchemann’s framework	1	122	2	0	
	2	113	1.99115	0.8914	
% students showing proceptual understanding	1	122	52.459	50.145	
	2	113	55.752	49.889	

Level 3 or above

Understanding of a letter at the level of at least a specific unknown.

Table 26: Full results for students initially at Küchemann’s level 3 or above

Analysis	Questionnaire	n	Mean	SD	p, if less than 0.05
Mean correct score	1	109	71.205	14.643	
	2	101	73.631	17.105	
Mean level on Küchemann’s framework	1	109	3.13761	0.34609	0.00105
	2	101	2.89109	0.7335	
% students showing proceptual understanding	1	109	96.33	18.8889	
	2	101	94.059	23.756	

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